ON COMPLETE INTERSECTIONS AND CONNECTEDNESS

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ABSTRACT

ON COMPLETE INTERSECTIONS AND CONNECTEDNESS

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In this thesis, we study the relation between connectedness and complete intersections. We describe the concept of connectedness in codimension k. We also study the basic facts about Cohen-Macaulay rings, and give some applications about these rings and complete intersections. Finally, we show that certain rational curves of degree ≥ 4 in the projective 3-space are set-theoretical complete intersections in all prime characteristics $p > 0$.

Keywords: Connectedness in codimension k, Cohen-Macaulay rings, set-theoretical and ideal-theoretical complete intersections, characteristic $p > 0$.

ÖZET

TAM KESİŞMELER VE BAĞLANTILILIK

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Bu tezde bağlantılılık ve tam kesişmeler üzerinde çalıştık. Eşboyutta bağlantılılık kavramını açıkladık. Cohen-Macaulay halkalarını inceleyip bunların tam kesişmeler ile ilişkisini açıklayan uygulamalar yaptık. Ayrıca, derecesi ≥ 4 olan bazı rasyonel eğrilerin projektif 3-uzayda tüm pozitif karakteristikler için tam kesişme olduklarını inceledik.

Anahtar kelimeler: Eşboyutta bağlantılılık, Cohen-Macaulay halkaları, ideal ve kümesel tam kesişmeler, karakteristik > 0.

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Contents

Chapter 1

Introduction

Complete intersections are one of the basic subjects of algebraic geometry. Many different relations between algebra and geometry can be found by understanding this subject. The purpose of this thesis is to describe a necessary topological condition for an algebraic set V to be a complete intersection. This condition is that V should be locally connected in codimension 1, which means that removing a subvariety which have codimension 2 or more does not make V disconnected. In order to achieve this result, we deal with the concepts of connectedness and codimension. Then we pass to the property of Cohen-Macaulay, which is a local property. Cohen-Macaulayness is also an important property in algebraic geometry which makes it possible to have links between geometry and algebra.

In chapter 2, we recall some topological concepts, such as connectedness. We describe the codimension of a variety. Then we explain the concepts of connectedness and locally connectedness in codimension k by giving the relations between them.

In chapter 3, we deal with Cohen-Macaulay rings. In order to understand these rings, we give the definitions of depth of an ideal (which is a measure of the size of the ideal), the height of an ideal, and R-sequences. We describe the Cohen-Macaulayness for local rings and then we generalize this definition. We also give a theorem of Hartshorne [6] on local rings, from which we get the necessary conditions for an algebraic set to be a complete intersection.

In chapter 4, complete intersections are studied in details. The definitions of set-theoretical and ideal theoretical complete intersections are given. We also study some basic examples of complete intersections in this chapter. We state the popular example *twisted cubic curve*, which is not a complete intersection in the strong sense (this means ideal theoretically), but a complete intersection set-theoretically. Then we restrict on Cohen-Macaulay rings and we use some basic facts about them to achieve our result by which we can make a connection between connectedness in codimension 1 and complete intersections. At the end of this chapter, we state the main theorem of Hartshorne about complete intersections and connectedness in codimension 1.

In chapter 5, we give various applications of this result. We study the following examples:

- An example of a subset $V \subset \mathbb{A}^4$ which is not a complete intersection.
- An example of an irreducible surface in $\mathbb{C}^4$ which is not a complete intersection.

- A counter example to the situation: If V is a complete intersection, then any two irreducible components of V passing through a given point P have an intersection which is of codimension 1 with respect to each of them at P.

- A remark of Serre: A complete intersection in the projective space is connected in codimension 1.

- An example of an irreducible 3-dimensional algebraic set V in $\mathbb{A}^5$ with $char(k) = 2$, such that V is a complete intersection set-theoretically, but the coordinate ring $k[V]$ is not Cohen-Macaulay.

In the last chapter, we show that certain rational curves of degree ≥ 4 in $\mathbb{P}^3$ are set-theoretically complete intersections in all prime characteristics $p > 0$. We state the other related results about characteristics $p > 0$ and give some examples. When we give the proof of the theorem of Hartshorne [7], we also give the procedure of finding the surfaces for complete intersections. We apply this procedure to some curves and find some surfaces. And then we observe that the intersection of these surfaces is the curve which we study. So we say that these curves are complete intersections set-theoretically.

We will use the following notations:

$\mathbb{A}^n$, for the affine $n-$space.

$\mathbb{P}^n$, for the projective $n-$space.

Chapter 2

Connectedness

In this chapter, we will give the definitions of connectedness and locally connectedness in codimension k. By giving this property here, we will be able to give a necessary condition for an algebraic set V to be a complete intersection. And this condition is about being locally connected in codimension one.

To give these definitions and some other concepts, we will report on Hartshorne's paper called Complete Intersections and Connectedness [6].

To introduce the concept of connectedness in codimension k, first let us understand the meanings of connectedness and codimension:

Definition 2.1 *Let A be a subset of a topological space X. We say that A is connected if A cannot be written as a union $A = B \cup C$ with*

$$B \neq \emptyset, \quad C \neq \emptyset$$

and

$$\overline{B} \cap C = B \cap \overline{C} = \emptyset,$$

where $\overline{B}$ and $\overline{C}$ are closures in X. Otherwise, A is called disconnected.

Definition 2.2 *Let X be a noetherian topological space and Y be an irreducible closed subspace of X. Then we define the codimension of Y in X to be the supremum of those integers n such that there exists a sequence of closed irreducible subspaces X_i of X,*

$$Y \subseteq X_0 \subset X_1 \subset \ldots \subset X_n \subseteq X.$$

And we denote it by $codim(Y, X)$.

In the affine category, the codimension of a subvariety Y of X becomes the number $dimX - dimY$. If Y is a subvariety of the affine space $\mathbb{A}^n$ of dimension $n-1$, then Y has codimension 1, and we say that Y is a hypersurface in $\mathbb{A}^n$.

If we look at the equations which define the varieties and the dimension of the variety, we see that each equation brings one restriction on the dimension of a variety, in general. So each variety defined by one equation has dimension one less than the space it belongs; that is, as we defined above, the variety has codimension one.

To give the definition of connectedness in codimension k, we should first give the following proposition from Hartshorne [6]:

Proposition 2.3 *Suppose that X is a noetherian topological space and k is an integer with $k \geq 0$. Then the following two conditions are equivalent:*

(i) If Y is a closed subset of X, and codim(Y, X)$>k$, then $X - Y$ is connected.

(ii) Let X' and X'' be irreducible components of X. Then we can find a finite sequence

$$X' = X_1, X_2, \ldots, X_n = X''$$

which is composed of irreducible components of X, such that for each $i = 1, 2, \ldots, n-1$, $X_i \cap X_{i+1}$ is of codimension $\leq k$ in X.

Proof : $(i) \Rightarrow (ii)$: Let us assume that (ii) does not hold, that is, there exists two irreducible components X' and X'' of X such that for every finite sequence of irreducible components

$$X' = X_1, X_2, \ldots, X_r = X'',$$

we have an index $i_0 \in \{1, 2, \ldots, r-1\}$ for which

$$codim(X_{i_0} \cap X_{i_0+1}, X) > k.$$

In particular, we can take $codim(X' \cap X'', X) > k$.

Let Y be the union of all subsets of X of the form $X_i \cap X_j$ where X_i, X_j s are the irreducible components of X and $codim(X_i \cap X_j, X) > k$. Then $codim(Y, X) > k$, and by (i), we have $X - Y$ is connected.

For any irreducible component X_i of X, let $Y_i = X_i \cap (X - Y)$. Since $X - Y$ is connected and noetherian, there exists a finite sequence

$$Y_1 = X' \cap (X - Y), \quad Y_2 = X_2 \cap (X - Y) \quad , \ldots, \quad Y_r = X'' \cap (X - Y),$$

such that $Y_i \cap Y_{i+1} \neq \emptyset$. But this gives s finite sequence $X_1, X_2, \ldots, X_r$ with $codim(X_i \cap X_{i+1}) \leq k$, which is a contradiction. This proves that (ii) holds when (i) holds.

$(ii) \Rightarrow (i)$: Suppose that (ii) is satisfied. Let Y be a closed subset of X with $codim(Y, X) > k$. Since for any irreducible components X_i, X_j of X, we must have $codim(X_i \cap X_j, X) \leq k$ by (ii), Y cannot contain any set of the form $X_i \cap X_j$. The irreducible components of $X - Y$ are of the form $X_i \cap (X - Y)$, where X_i is an irreducible component of X.

Let $p \neq q$ be two points of $X - Y$. Let $p \in X'$, $q \in X''$ where X' and X'' are irreducible components of X. Then there is a finite sequence of irreducible components of X,

$$X' = X_1, X_2, \ldots, X_r = X''$$

such that $codim(X_i \cap X_{i+1}, X) \leq k$, for some $i = 1, 2, \ldots, r-1$. Letting $Y_i = X_i \cap (X - Y)$, for some $i = 1, 2, \ldots, r$, we have $Y_i \cap Y_{i+1} \neq \emptyset$, for $i = 1, 2, \ldots, r-1$. This shows that $X - Y$ is connected. □

Now, we can define connectedness in codimension k:

Definition 2.4 *Let X be a noetherian topological space, and $k \geq 0$ be an integer. If X satisfies any of the equivalent conditions of the above proposition, we say that X is connected in codimension k.*

Geyer [5], also gives a similar definition in his lecture notes. He says that a noetherian topological space X of finite dimension (so especially an algebraic set) is connected in codimension one, if $X - Y$ is connected for every closed subset Y with

$$1 + dimY < dimX.$$

We can see that, this easily follows from the definition of codimension. From Definition 2.2, we have

$$codim(Y, X) = dimX - dimY,$$

and from the above definition of connectedness in codimension k, that is from Proposition 2.3 (i), we have

$$codim(Y, X) > k.$$

So Geyer's definition of connectedness in codimension $1 = k$ easily follows from

$$codim(Y, X) = dimX - dimY > 1 = k,$$

hence

$$dimX > 1 + dimY.$$

From the definition of connectedness in codimension k, we can get the following results:

Remark 2.5 (i) *If X is connected in codimension k (for any $k \geq 0$), then X is connected.*

Proof : Let X be connected in codimension k. Then there exists a closed subset $Y \subset X$ with $codim(Y, X) > k$ such that $X - Y$ is connected. If we suppose that X is not connected, then we can write X as a union of two sets $X = B \cup C$ such that

$$B \neq \emptyset, \quad C \neq \emptyset$$

and

$$\overline{B} \cap C = B \cap \overline{C} = \emptyset.$$

Now we have

$$X - Y = (B \cup C) - Y = (B - Y) \cup (C - Y),$$

but this contradicts with $X - Y$ s being connected. Hence X must be connected. $\square$

(ii) If X is connected in codimension k, then for any $l \geq k$, X is also connected in codimension l.

We will give another remark now which will be used in the next chapter to prove Theorem 3.9.

Remark 2.6 *If a noetherian topological space X is connected in codimension 1, then the following conditions are satisfied:*

(i) X is connected. This easily follows from Remark 2.5 (i).

(ii) The irreducible components X_i of X all have the same dimension as X.

This can be proved as follows: Let us take $dimX = r$. If we suppose that $dimX_i < dimX = r$, for some irreducible component X_i, then we can find a subset

$$Y = X_i \cap (\cup X_j)$$

with $i \neq j$ where X_j s are the other irreducible components of X. Since

$$dimY < dimX_i,$$

we have that

$$1 + dimY \leq dimX_i < dimX.$$

But now, we find that $X - Y$ is not connected; points in X_i which do not lie in any other component form a set which is disjoint in $X - Y$ from the other components. This contradicts with our assumption.

(iii) If X' and X'' are any two irreducible components of X, then we can find a chain

$$X' = W_0, W_1, \ldots, W_s = X''$$

such that W_k s are all irreducible components of X for $k = 0, \ldots, s$ satisfying

$$dim(W_k \cap W_{k+1}) = r - 1.$$

Conversely, if the above condition *(iii)* is satisfied, then the noetherian space X of dimension r is connected in codimension 1. This is clear from Definition 2.4, since

$$codim(W_k \cap W_{k+1}, X) = dimX - dim(W_k \cap W_{k+1}) = r - (r - 1) = 1.$$

Now we will give the definition of locally connectedness in codimension k. But before giving this definition, we should give the following concepts:

Definition 2.7 *If Y is a closed irreducible subset of the topological space X, $y \in Y$ is called a generic point for Y if Y is the closure of the set consisting of the single point y.*

In the next chapter, we will deal with the spectrum of a ring. So let us mention here the generic point in the spectrum of a ring. If R is a ring, then a point x which is everywhere dense in $Spec(R)$ is said to be generic. For instance, if we take the ring $\mathbb{Z}$ of integers, the point $Spec(\mathbb{Z})$, which corresponds to the zero ideal of $\mathbb{Z}$, is generic: that is, the ideal (0) is prime and is contained in every prime ideal , therefore its closure is the whole space and it is an everywhere dense point.

Definition 2.8 *A topological space which has a single closed point is called a local topological space. Let X be any topological space and $y \in X$. The local space of X at the point y is defined to be the set of points z whose closures contain y. This set is called the generalizations of y. And this local space of X at y is denoted by X_y.*

Lemma 2.9 *(Hartshorne [6]) Suppose that X is a connected topological space and Y is a closed subspace of X such that for every $y \in Y$, $X_y - y$ is non-empty and connected. Then $X - Y$ is connected.*

Proof: See [6], page 499.

Proposition 2.10 *Suppose that X is a noetherian topological space, and k is an integer with $k \geq 0$. Then the following conditions are equivalent:*

(*i*) *The local space X_y is connected in codimension k, for any $y \in X$.*

(ii) If $y \in X$ *is such that dim* $X_y > k$*, then* $X_y - y$ *is connected.*

Proof : $(i) \Rightarrow (ii)$: Let X_y be connected in codimension k for any $y \in X$, then there exists a closed subset Y of X_y such that $codim(Y, X_y) > k$, and $X_y - Y$ is connected. Then we have $X_y - y$ is connected for any $y \in X$.

$(ii) \Rightarrow (i)$: Let $y \in X$ and Y be a closed subset of X_y, with $codim(Y, X_y) > k$. Then for any $z \in Y$, we have $dimX_z > k$. Therefore, from (ii), $X_z - z$ is connected. And since $dimX_z > k \geq 0$, $X_z - z$ is non-empty. But since the local space X_y is connected, we can now apply the above lemma. Here X_y is our connected local space and Y is the closed subset of X_y. We have $X_z - z$ is non-empty and connected, for any $z \in Y$. So all conditions of the lemma are satisfied. Hence, we say $X_y - Y$ is connected. So by the definition, X_y is connected in codimension k. □

Now, we can give the definition:

Definition 2.11 *Suppose that* X *is a noetherian topological space and* X *satisfies any of the equivalent conditions of the above proposition for any integer* $k \geq 0$*. Then* X *is said to be locally connected in codimension* k*.*

Similar to the properties of connectedness in codimension k, we can state the following:

Remark 2.12 *(i) If* X *is locally connected in codimension* k*, then for any* $l \geq k$*,* X *is locally connected in codimension* l*.*

(*ii*) *If X is connected and locally connected in codimension k, then it is connected in codimension k. Let us sketch the proof shortly:*

Suppose that X is connected and locally connected in codimension k. Let us take $y \in X$ such that $dim X_y > k$. Then from Proposition 2.10, we have that $X_y - y$ is connected. If we take a closed subset Y of X, then from Lemma 2.9, it follows that $X - Y$ is connected. Hence, from Proposition 2.3, we say that X is connected in codimension k.

The converse of this statement is not true. (See Hartshorne [6], page 500).

(*iii*) *For a local topological space X, being locally connected in codimension k implies being connected in codimension k.*

Proof : If X is a topological space which is locally connected in codimension k, then by Proposition 2.10, for a closed subspace Y of X, and for every $y \in X$ satisfying $dim X_y > k$, we have $X_y - y$ is connected. Then it follows from Lemma 2.9 that $X - Y$ is connected. And by Proposition 2.3 and Definition 2.4, we say that X is connected in codimension k.

Chapter 3

Depth and Cohen-Macaulay Rings

In this chapter, we will try to describe the facts about Cohen-Macaulay rings. These rings are important, because by studying them many relations between algebra and geometry are observed.

Our aim in giving Cohen-Macaulay rings is that in the next chapter, when we study the concept of complete intersections, we will restrict ourselves to Cohen-Macaulay rings to avoid some difficulties.

Cohen-Macaulay rings can be described in many different ways, by many different approaches. Eisenbud [3] describes these rings as rings R in which

$$depth(I, R) = codim I$$

for every ideal I (it is enough to assume this when I is a maximal ideal). The concept of $codim I$ is explained below.

Let us take the ring R, and let $I \subset R$ be an ideal of R with $I \neq R$. The *dimension* of I, denoted by $dim I$, is defined to be $dim R/I$. If I is a prime ideal then the *codimension* of I, denoted by $codim I$, (also called height I and rank I by various authors), is defined to be the dimension of the local ring R_I. In other words, $codim I$ is the supremum of lengths of chains of primes descending from I. If I is not a prime ideal, then $codim I$ is the minumum of the codimensions of the primes containing I, see Definition 3.5.

We will give a definition of Cohen-Macaulay rings which depends on the depth of ideals in a ring. Before we can define depth, we need some definitions:

Definition 3.1 *Let R be a noetherian ring. A sequence $r_1, r_2, \ldots, r_n$ of elements in R is called an R-sequence or a regular sequence on R if it satisfies the following properties:*

(i) $R \neq (r_1, r_2, \ldots, r_n)R$

(ii) For $i = 1, \ldots, n$, the i th element r_i is not a zero divisor on $R/(r_1, r_2, \ldots, r_{i-1})R$.

Example The basic example of a regular sequence is the sequence $x_1, \ldots, x_n$ of indeterminates in a polynomial ring $R = k[x_1, \ldots, x_n]$.

Example Let us consider the ring

$$R = k[x, y, z]/(x - 1)z,$$

and the sequence of elements $x, (x-1)y$. The ideal generated by these two elements is

$$(x, (x-1)y) = (x, y) \neq R.$$

And x is not a zero-divisor in R, and $R/(x) = k[y, z]/(z)$. So following the definition, we see that the sequence $x, (x-1)y$ is a regular sequence.

Eisunbud [3] uses *Koszul complex* to show the basic properties about depth. He mentions that depth is an algebraic concept which is parallel to the geometric concept of codimension. And his approach to the Cohen-Macaulayness is in a way such that it is a condition in which these two concepts in algebra and geometry are given together.

Let us now give the definition of depth:

Definition 3.2 *Let R be a noetherian ring, and I be an ideal in R. The length of any maximal R-sequence in I is defined to be the depth of an ideal I.*

As we see in the definition, the depth of I is a measure of the size of I, while the codimension of I is a geometric measure.

Instead of depth, the terms *homological codimension*, and *grade* is also used.

It is necessary here to mention that all maximal R-sequences have the same length (see [15], volume II, appendix 6).

We should also mention here a property of depth which we will use in the examples in chapter 5. But first, let us give the definition of the annihilator:

Definition 3.3 *Let R be a ring, and M be a module of R. The set of elements r of R such that* $rM = (0)$ *is called the annihilator of M and is denoted by* $ann(M)$*, that is,*

$$ann(M) = \{r \in R : rM = 0\}.$$

If P is an associated prime ideal of R, that is, a prime ideal which is the annihilator of some non-zero element of R, then we have

$$depth(R) \leq dimP$$

which implies

$$depth(R) \leq dimR < \infty.$$

(See Geyer [5], page 232).

Now, we will give the definition of a Cohen-Macaulay ring. In the first case, we will define Cohen-Macaulay rings for local rings. A ring is *local*, if it is noetherian and has exactly one maximal ideal. If the ring R is local, and M is the unique maximal ideal of it, we denote this by saying (R, M) is a local ring.

Definition 3.4 *Let R be a local ring. R is called a Cohen-Macaulay ring, if the depth of R is equal to its dimension.*

For some other approaches to Cohen-Macaulay rings one can refer to [9].

Example The polynomial ring $A = k[x_1, \ldots, x_n]$ over a field k is a Cohen-Macaulay ring.

Example Any regular local ring A is a Cohen-Macaulay ring. (We give the definition of a regular local ring in page 28).

Example Any local domain A of dimension 1 is a Cohen-Macaulay ring. Indeed, any single element $\neq 0$ of the maximal ideal of A constitutes a prime sequence. (An example from [15], volume II, page 397).

Let us now define the height of an ideal:

Definition 3.5 *In a ring R, the height of a prime ideal P is the supremum of all integers n such that there exists a chain*

$$P_0 \subset P_1 \subset \ldots \subset P_n = P$$

of distinct prime ideals. The height of any ideal I is the infimum of the heights of the prime ideals containing I.

When we give the definition of Cohen-Macaulay rings above, we defined it for local rings. Let us now state the definition in a more general way.

We say that a noetherian ring R is a Cohen-Macaulay ring, if for every maximal ideal P of R, the localization R_P is Cohen-Macaulay with dimension

$$dim R_P = dim R.$$

Then we have the equation

$$dim P + height P = dim R$$

for all $P \in Spec(R)$, where $dimP$ is defined as $dimR/P$. (See Eisenbud [3], page 225, or [1]).

Remark 3.6 *Here, by Spec(R), we denote the set of all prime ideals in the ring R.*

For instance, if we take the ring $\mathbb{Z}$ of integers, then we see that $Spec(\mathbb{Z})$ consists of the ideal (0) and the ideals (p) for all prime numbers p.

Now, let us define the Zariski topology on $Spec(R)$. Let R be any ring and I be an ideal of R. The subsets

$$Z(I) = \{P : P \text{ is a prime ideal of } R \text{ and } P \supset I\} \subseteq Spec(R)$$

are called Zariski closed subsets (shortly closed subsets). Since any finite unions and arbitrary intersections of closed subsets are closed, the closed subsets define a topology on $Spec(R)$, and this topology is called the Zariski topology.

Remark 3.7 *Geyer [5] gives the following properties of Cohen-Macaulay rings, which we will use in the following chapters:*

If R is a Cohen-Macaulay ring, then the following are all Cohen-Macaulay:

(i) R_P, for every $P \in Spec(R)$;

(ii) the completion R^*, if R is local;

(iii) R/Rx*, if* x *is not a zero-divisor.*

A ring R is Cohen-Macaulay iff the polynomial ring $R[x]$ is Cohen-Macaulay (for the proof see [3], page 452).

The following theorem is important, since it is related with our main results. We will also use the following theorems in the next two chapters.

Theorem 3.8 *(Hartshorne, [6]). Suppose that* (R, M) *is a local ring and* $Spec(R) - M$ *is disconnected in Zariski topology. Then we have*

$$depth R = 1.$$

Proof : See [5], page 234.

By using the concepts of connectedness in codimension 1, and Remark 2.6, we can restate the above theorem for Cohen-Macaulay rings as follows:

Theorem 3.9 *(Hartshorne, [6]). Suppose that* R *is a Cohen-Macaulay ring, and* R *cannot be written as a direct product of rings. Then* $X = Spec(R)$ *is connected in codimension 1.*

Proof : (Geyer [5], page 237).

Chapter 4

Complete Intersections

In this chapter we will look deeper into algebraic geometry. We will introduce the concept of complete intersections. This is a very basic subject of algebraic geometry and is related with different subjects of algebra and geometry.

We will first give the definitions of set-theoretically and ideal theoretically complete intersections. And then we will describe the relation between connectedness and complete intersections. Finally, we will give a necessary topological condition for an algebraic set V to be complete intersection. In the next chapter, we will study some of the applications of this result.

We begin with the basic definitions:

Let k be an algebraically closed field of characteristic 0. If f is a nonzero homogeneous polynomial in $k[x_0, \ldots, x_n]$, then algebraic varieties of the form

$$\{f = 0\} \subset \mathbb{P}^n$$

are called *hypersurfaces*. A hypersurface H in $\mathbb{P}^n$ has dimension $n-1$.

An algebraic set V in $\mathbb{P}^n$ is the set of common zeros of a subset S of homogeneous polynomials in the polynomial ring $k[x_0, \ldots, x_n]$; that is,

$$V(S) = \{P \in \mathbb{P}^n : f(P) = 0, \quad \text{for all} \quad f \in S\}.$$

And if we take any subset V of $\mathbb{P}^n$; we define the ideal of V as follows:

$$I(V) = \{f \in k[x_0, \ldots, x_n] : \text{f is homogeneous and} \quad f(P) = 0, \quad \text{for all} \quad P \in V\}.$$

(For details, see [8], chapter 1).

Definition 4.1 *An algebraic set V (whose every irreducible component has dimension $n-r$) is called a set-theoretically complete intersection, if it is the intersection of r hypersurfaces $\{f_i = 0\}$ in the n-space; that is,*

$$V = V(f_1, \ldots, f_r).$$

In other words, a variety V of dimension r in the projective space $\mathbb{P}^n$ is a set-theoretically complete intersection, if there are $n - r = codim(V, \mathbb{P}^n)$ hypersurfaces whose intersection is V.

Moreover, if the f_i s can be chosen so that

$$I(V) = (f_1, \ldots, f_r),$$

then we say that V is ideal-theoretically a complete intersection.

Every ideal theoretical complete intersection is a set-theoretical complete intersection, but the converse does not hold as we will demonstrate later.

We can say that complete intersections are those algebraic sets which can be described by the lowest possible number of equations. Let us give some examples of complete intersections:

Example The trivial algebraic sets $\mathbb{A}^n$ and $\mathbb{P}^n$ are ideal theoretically complete intersections . Because, $\mathbb{A}^n$ and $\mathbb{P}^n$ are both the intersection of 0 hypersurfaces.

Example Every point is ideal theoretically a complete intersection. If we take the point

$$P = [p_0 : p_1 : \ldots : p_n] \in \mathbb{P}^n,$$

and assume that $p_j \neq 0$ for some j, then in $k[x_0, \ldots, x_n]$ we can write P as the intersection of n hyperplanes

$$p_j X_i - p_i X_j = 0 \quad for \quad (i = 0, \ldots, n \quad and \quad i \neq j).$$

And the ideal $I(P)$ is generated by these polynomials.

Let us now give an example which includes basic facts about complete intersections.

Example (i) Let X be a variety in the projective space $\mathbb{P}^n$, and suppose that $X = Z(I)$ and I can be generated by r elements. Then let us see that $dimX \geq n - r$.

Since I is generated by r elements, we can take

$$I = (f_1, \ldots, f_r),$$

where f_i s are homogeneous polynomials in $k[x_0, \ldots, x_n]$ for $i = 1, \ldots, r$. And since $X = Z(I)$, where $Z(I)$ is the zero set of I, then we have

$$X = Z(I) = Z(f_1, \ldots, f_r) \subseteq \mathbb{P}^n.$$

A hypersurface $\{f_1 = 0\} \subset \mathbb{P}^n$ has dimension $n - 1$, so we get

$$dim Z(f_1) = n - 1,$$

$$dim Z(f_1, f_2) \geq n - 2.$$

If we continue in this way, we find that

$$dim Z(f_1, \ldots, f_r) \geq n - r.$$

Hence, we have

$$dim X \geq n - r.$$

(ii) If Y is a complete intersection ideal theoretically (it is also said that it is a complete intersection in the strong sense) with dimension r in $\mathbb{P}^n$, then Y is a complete intersection set-theoretically.

It is easy to show this: Since Y is a complete intersection in the strong sense with dimension r, then we have the ideal

$$I(Y) = (f_1, \ldots, f_{n-r})$$

generated by $n - r = codim(Y, \mathbb{P}^n)$ elements. Then we can write

$$Y = Z(f_1) \cap \ldots \cap Z(f_{n-r});$$

that is, we can write Y as an intersection of $n - r$ hypersurfaces. So Y is set-theoretically a complete intersection.

(iii) The converse of the above statement (ii) is not true. That is, if Y is set-theoretically a complete intersection, then it may not be a complete intersection in the strong sense.

We can give the twisted cubic curve as an example of this situation: Let us suppose that $Y \subseteq \mathbb{P}^3$ is the set

$$Y = \{[s^3 : s^2t : st^2 : t^3] \;\; : \;\; s, t \in k \quad and \quad (s,t) \neq (0,0)\}.$$

Here, Y is a variety of dimension 1. And if we take

$$\begin{aligned} x &= s^3 \\ y &= s^2t \\ z &= st^2 \\ w &= t^3, \end{aligned}$$

then it is easily seen that the ideal $I(Y)$ is generated by the polynomials

$$f_1 = z^2 - yw$$

$$f_2 = y^2 - xz$$

$$f_3 = xw - yz, 0$$

that is,

$$I(Y) = (f_1, f_2, f_3).$$

Here, it can be shown that the ideal $I(Y)$ cannot be generated by $2 = codim(Y, \mathbb{P}^3)$ polynomials, so Y is not a complete intersection in the strong sense. But, we can find two hypersurfaces

$$H_1 = Z(z^2 - yw)$$

and

$$H_2 = Z(y^3 - 2xyz + x^2w)$$

such that $Y = H_1 \cap H_2$. And hence Y is a complete intersection set-theoretically with these hypersurfaces.

Example We give an example of a set-theoretical complete intersection in affine space which is not a complete intersection ideal theoretically. This is an irreducible algebraic set V of dimension 1 in the affine 3-space $\mathbb{A}^3$. Let V be as follows:

$$V = \{(t^3, t^4, t^5) \in \mathbb{A}^3 \ : \ t \in k\}.$$

V has only one singularity, and that is the origin $(0, 0, 0)$. This is indeed a monomial space curve with

$$x_1 = t^3,$$

$$x_2 = t^4,$$

$$x_3 = t^5.$$

If we take the polynomials

$$f_1 = x_2^2 - x_1x_3,$$

$$f_2 = x_1^3 - x_2x_3,$$

$$f_3 = x_3^2 - x_1^2x_2,$$

then it can be shown that the prime ideal I(V) is generated by these polynomials; that is

$$I(V) = (f_1, f_2, f_3).$$

Here, we should mention that $I(V)$ cannot be generated by 2 polynomials. (We can show this by considering the degrees: Suppose that $I(V)$ is generated by 2 elements, that is, let

$$I(V) = (g_1, g_2).$$

Let us take

$$deg(x_1) = deg(t^3) = 3,$$

$$deg(x_2) = deg(t^4) = 4,$$

$$deg(x_3) = deg(t^5) = 5;$$

so we find that

$$deg(f_1) = 8,$$

$$deg(f_2) = 9,$$

$$deg(f_3) = 10.$$

If $deg(g_1) > 8$, and $deg(g_2) > 8$, then $I = (g_1, g_2)$ cannot generate $f_1 = x_2^2 - x_1x_3$. So either g_1 or g_2 should have degree 8, without loss of generality we can take $deg(g_1) = 8$. It is easily seen that, a monomial $x_1^{\alpha}x_2^{\beta}x_3^{\gamma}$ has degree at least 3, and we see that a polynomial contained in the ideal generated by g_1 has degree at least 11. But now the ideal generated by g_1 does not include the elements f_2 and f_3. So let us choose $deg(g_2) = 9$, then we cannot obtain the element f_3. So the ideal $I(V)$ cannot be generated by 2 elements.)

But V is set-theoretically a complete intersection, since it can be written as the intersection of $2 = codim(V, \mathbb{A}^3)$ surfaces

$$V = H_1 \cap H_2,$$

where

$$H_1 : g_1 = x_3^2 - x_1^2 x_2 = 0$$

$$H_2 : g_2 = x_1^4 + x_2^3 - 2x_1 x_2 x_3 = 0.$$

We will now give the definition of locally complete intersections:

Definition 4.2 *An algebraic set V is said to be locally a complete intersection, if every point of V has a neighborhood in which V is a complete intersection.*

In the above example, we see that

$$V = \{(t^3, t^4, t^5) \in \mathbb{A}^3 \ : \ t \in k\} \subseteq \mathbb{A}^3$$

is a set-theoretical complete intersection. But since $I(V)$ cannot be generated by 2 elements, V is not a complete intersection ideal theoretically. And V is not a local complete intersection.

The twisted cubic curve (given in page 25) is a local complete intersection, since we can write one of the generators f_1 as a combination of the others f_2 and f_3, when x is set to 1 for example.

Let us consider a local ring (R, M) with dimension r. By the Principal Ideal Theorem (see [3], page 232), it follows that M cannot be generated by less than r elements. And by Krull's Principal Ideal Theorem (see [11]), we say that R is called a *regular local ring* if M can be generated by exactly r elements.

Definition 4.3 *We call that a local ring R of dimension r is a complete intersection, if it can be written as the quotient of a regular local ring B by an ideal I which can be generated by $dim B - r$ elements.*

It can be seen from the Cohen-Macaulay theorem ([15], volume II, page 397, Theorem 2), that if $A = B/I$ is a complete intersection, then I can be generated by a $B-$sequence, as in the definition, and A is a Cohen-Macaulay ring.

The theorem we will state below is a basic theorem on which the general concept of complete intersections in noetherian rings are based:

Theorem 4.4 *(Krull's principal ideal theorem). Suppose that I is an ideal in the noetherian ring R, and P is a minimal prime ideal containing I. If $height(P) = r$, then I needs at least r generators as $R-$ideal.*

For the proof see [10], page 110.

As we mentioned before, we will define complete intersections to be ideals which can be generated by the lowest possible number of polynomials or the algebraic sets which can be described by the lowest possible number of equations according to the above theorem.

From now on, we will restrict ourselves on Cohen-Macaulay rings to avoid some difficulties.

Definition 4.5 *Let R be a Cohen-Macaulay ring. We say that the ideal I is a complete intersection (or equivalently we say that r elements $f_1, \dots, f_r$*

of R generate a complete intersection $I = (f_1, \ldots, f_r)$) if all minimal prime ideals P containing I are of maximal height $= r$ with $I \neq R$.

I is called locally a complete intersection at $P \in Spec(R)$, if I_P is a complete intersection in R_P.

Proposition 4.6 *Suppose that R is a Cohen-Macaulay ring, and $f_1, \ldots, f_r \in R$. Then the following two conditions are equivalent:*

(i) $f_1, \ldots, f_r$ generate a complete intersection $I = (f_1, \ldots, f_r)$.

(ii) $f_1, \ldots, f_r$ is an R−sequence.

Proof : See [5], page 239.

Proposition 4.7 *Suppose that I is a complete intersection in the Cohen-Macaulay ring R. Then R/I is also a Cohen-Macaulay ring.*

The proof of this proposition is contained in the above proof of Proposition 4.6. Let us now state the proof in another way:

Proof : Let us take $I = (r_1, \ldots, r_n)$ in R which is generated by $n = codimI$ elements, where n is the largest possible value. This means that I is a complete intersection. R is Cohen-Macaulay iff R_P is Cohen-Macaulay for every prime ideal P of R. So we may assume that R is a local ring with maximal ideal M. If we choose a maximal regular sequence in M which begins with $(r_1, \ldots, r_n)$, then we have

$$depth(R/I) = depthR - n.$$

For $i = 1, \ldots, n$, the i th element r_i is not in any of the minimal primes of $(r_i, \ldots, r_{i-1})$, so we get

$$dim(R/I) \leq dimR - n.$$

And since R is a Cohen-Macaulay ring, it implies that

$$dim(R/I) \leq dimR - n = depthR - n = depth(R/I),$$

and since we always have

$$depth(R/I) \leq dim(R/I),$$

it follows that

$$depth(R/I) = dim(R/I).$$

So we conclude that R/I is Cohen-Macaulay. □

Remark 4.8 *For a Cohen-Macaulay ring R and a closed subset $X \subset Spec(R)$; we have the following:*

X is a set-theoretical complete intersection in $Spec(R)$, if $X = V(I)$, where I is a complete intersection. This means that for some $f_i \in R$ and for all irreducible components of X which have codimension r in $Spec(R)$, we have $X = V(f_1, \ldots, f_r)$, where

$$V(f_1, \ldots, f_r) = \{P \in Spec(R) : (f_1, \ldots, f_r) \subset P\}.$$

Here we will shortly mention the concept of a coordinate ring which will be used in the next chapter, in Example 5.5. Now, let us suppose that

$$f_1 = \ldots = f_r = 0$$

define a set-theoretically complete intersection, say V, in $\mathbb{A}^n$. Since $(f_1, \ldots, f_r)$ is a complete intersection in the Cohen-Macaulay ring $k[x_1, \ldots, x_n]$, from Proposition 4.7, we have that

$$\overline{R} = k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$$

is also a Cohen-Macaulay ring. The radical ideal associated to the algebraic set V is

$$I(V) = rad(f_1, \ldots, f_r).$$

And the coordinate ring associated to V is

$$k[V] = k[x_1, \ldots, x_n]/I(V) = \overline{R}/rad(0).$$

In general when I is not a complete intersection ideal, the coordinate ring $k[V]$ of V may not be a Cohen-Macaulay ring, although $k[x_1, \ldots, x_n]$ is Cohen-Macaulay. We will study an example about a coordinate ring which is not Cohen-Macaulay in the next chapter (in chapter 5, Example 5.5).

Here, we will state a property about completions for complete intersections:

Proposition 4.9 *Let the local ring A be a complete intersection, then the completion A^* of A is also a complete intersection.*

Proof : From Definition 4.3, let us write A as a quotient B/I, where B is a regular local ring, and I can be generated by $dimB - dimA$ elements. If we take completions, we have

$$A^* = B^*/I^*$$

since completion is an exact functor (see [15], volume II, chapter VIII, § 2). Now, since completion preserves the dimension and the regularity of a local ring (see [15], volume II, chapter VIII, § 11); we have

$$dimA = dimA^*,$$

$$dimB = dimB^*,$$

and B^* is regular. Moreover, we have

$$I^* = IB^*.$$

So I^* can be generated by

$$(dimB) - (dimA) = (dimB^*) - (dimA^*)$$

elements. So we see that A^* is written as the quotient of a regular ring B^* by an ideal I^*, which is generated by $dim(B^*) - dim(A^*)$ elements. So we find that A^* is a complete intersection. □

Now, we will pass to the local conditions for complete intersections and finally give the main theorem. We will use Proposition 4.7 stated above and Theorems 3.8, 3.9 stated in the previous chapter.

Theorem 4.10 *(Hartshorne, [6]). Suppose that (R, M) is a local Cohen-Macaulay ring. Let $V = V(I)$ be a set-theoretical complete intersection in $Spec(R)$ with dimension > 1. Then $V - \{M\}$ is connected. And $V^* - \{M^*\}$ is also connected.*
Here $V^* = Spec(R^*/I^*)$ is called the completion of V at R.

Proof : Since R is a Cohen-Macaulay ring and $V = V(I)$ is a set-theoretical complete intersection, then by Proposition 4.7, we have that R/I is Cohen-Macaulay. It follows from Proposition 4.9 that the completion

$$(R/I)^* = R^*/I^*$$

is also a Cohen-Macaulay ring. Here R/I and R^*/I^* both have dimension bigger than 1. Since they are both Cohen-Macaulay rings, we have that

$$depth(R/I) = dim(R/I)$$

and

$$depth(R/I)^* = dim(R^*/I^*).$$

So the depth is also bigger than 1. By the Theorem 3.8 in the previous chapter, since we have

$$depth(R/I) > 1$$

and

$$depth(R^*/I^*) > 1,$$

it follows that both $V - \{M\}$ and $V^* - \{M^*\}$ are connected. □

We will use the above theorem in Example 5.6, in chapter 5.

As a result of the above theorem, we give the following:

Corollary 4.11 *Let the algebraic set $V \subset \mathbb{A}^n$ be locally a set-theoretical complete intersection at a point $P \in V$ and suppose that $codim(P, V) > 1$. Then $V - \{P\}$ is connected locally at P.*

Finally, we can state the main theorem:

Theorem 4.12 *(Geyer [5]). Suppose that V is an algebraic set which is locally a complete intersection everywhere. Then the connected components of V are connected in codimension 1.*

Proof : ([5], page 242). If we take the affine ring $R = k[x_1, \ldots, x_n]/I$ with a connected spectrum, since the Cohen-Macaulayness condition is local, the result follows from Theorem 3.9. And for the projective case, it follows in the same way by Theorem 3.9. $\square$

This theorem is stated for preschemes in Hartshorne's paper [6].

By this theorem we get a necessary local condition for an algebraic set to be a complete intersection, and this condition is that V should be connected in codimension 1.

In chapter 5, we will study an example which is not a complete intersection in Example 5.1 and 5.6.

Chapter 5

Examples

In this chapter, we will give some applications of the things we have done in the previous chapters. We will study examples about various complete intersections and their connectedness in codimension k, Cohen-Macaulay rings, some surfaces which are not complete intersections, and an example about an irreducible algebraic set V whose coordinate ring $k[V]$ is not Cohen-Macaulay.

Example 5.1 We will see that the subset of affine 4-space $X = \mathbb{C}^4$ consisting of two planes which meet in a single point is not a complete intersection in this example of Hartshorne [6]. Let V be the variety which is the union of two planes

$$z_1 = z_2 = 0$$

and

$$z_3 = z_4 = 0.$$

Now, let us show that V is not a complete intersection.

From Theorem 4.12, to say that V is not a complete intersection, it is enough to show that V is not connected in codimension 1. By the definition of connectedness in codimension 1; let us take

$$k = 1 > 0, \quad Y = \{(0,0,0,0\}).$$

We have that $dimV = 2$, $codim(Y,V) = 2 > 1$, and

$$V - Y = V - \{(0,0,0,0\})$$

is not connected by Definition 2.4. So we say that V is not connected in codimension 1. Hence, we conclude that V is not a complete intersection.

Example 5.2 Let us find an irreducible surface in $X = \mathbb{C}^4$ which is not a complete intersection. Suppose that V is a surface given parametrically by

$$\begin{aligned} z_1 &= t \\ z_2 &= tu \\ z_3 &= u(u-1) \\ z_4 &= u^2(u-1), \end{aligned}$$

where $(t,u) \in \mathbb{C}^2$.
It is easily seen that the points $(0,0)$ and $(0,1)$ are both mapped into the origin $(0,0,0,0)$ of X. Since V is irreducible, it is locally connected in codimension 1. But at the origin V is not formally connected in codimension 1 (from Proposition 2.3). So by Theorem 4.12, we say that V is not a complete intersection. This example shows that two hypersurfaces are not sufficient

to define an irreducible surface V in 4-space.

Example 5.3 One expects that if V is a complete intersection, then any two irreducible components of V passing through a given point P have an intersection which is of codimension 1 with respect to each of them at P, but we have a counter example to this situation: Take $X = \mathbb{C}^4$, and let V be the union of three planes Q_1, Q_2, Q_3. Suppose that

Q_1 is defined by $z_1 = z_2 = 0$,

Q_2 is defined by $z_2 = z_3 = 0$, and

Q_3 is defined by $z_3 = z_4 = 0$.

We can write $V = V(f_1, f_2)$ where

$$f_1 = z_1 z_3 + z_2 z_4,$$

and

$$f_2 = z_2 z_3,$$

that is we write V as the intersection of $2 = codim(V, X)$ hypersurfaces. So V is a complete intersection. But the two components Q_1 and Q_3 of V meet only in a point $(0, 0, 0, 0)$, and this point is of codimension 2, not 1 in V.

Example 5.4 Now, we will study a remark of Serre given in [5], page 242. We will show that a complete intersection in projective space $\mathbb{P}^n$ is connected in codimension 1.

We state the proof given by Geyer. There is also another proof given by Hartshorne [6], page 508.

Suppose that V is a complete intersection in $\mathbb{P}^n$, and V is not connected codimension 1. Then from Definition 2.4, we can say that there exists a closed subset

$$W \subset V = V(f_1, \ldots, f_r)$$

with

$$codim W > 1 = k$$

such that $V - W$ is disconnected. Then there exists a linear subspace

$$L \subset \mathbb{P}^n$$

with

$$dim L = r + 1$$

given by linear equations

$$f_{r+1} = \ldots = f_{n-1} = 0,$$

satisfying $L \cap W = \emptyset$. But L meets all irreducible components of V by a theorem given in Geyer ([5], page 204, Theorem 4). Then we have

$$V(f_1, \ldots, f_r, f_{r+1}, \ldots, f_{n-1}) \subset V - W,$$

and since $V - W$ is disconnected, $V(f_1, \ldots, f_r, f_{r+1}, \ldots, f_{n-1})$ is also disconnected. But this is a contradiction. Because we have $n-1 < n$, so by Geyer's theorem V should be connected. But here we find that V is disconnected. So if V is a complete intersection in $\mathbb{P}^n$, then it must be connected in codimension 1.

Example 5.5 (An example of Geyer ([5], page 244).) Now, we will see an example of an irreducible 3-dimensional algebraic set V in $\mathbb{A}^5$ with $char(k) = 2$, such that V is a complete intersection set-theoretically, but the coordinate ring $k[V]$ is not Cohen-Macaulay.

First let us make some preparations:

The subring

$$T = k[\xi^2, \eta, \mu, \xi\eta, \xi\mu]$$

of the polynomial ring $k[\xi, \eta, \mu]$ in 3 variables can be given by

$$T \cong k[X_1, X_2, X_3, X_4, X_5]/P$$

where

$$P = (X_4^2 - X_1X_2^2, X_5^2 - X_1X_3^2, X_5X_2 - X_3X_4, X_5X_4 - X_1X_2X_3).$$

Now, let's look at the factor ring

$$\overline{T} = T/Tx_2 \cong k[X_1, X_3, X_4, X_5]/(X_4^2, X_5^2 - X_1X_3^2, X_3X_4, X_4X_5).$$

We have that

$$Ann(x_4) = (x_3, x_4, x_5),$$

since if $f \in Ann(x_4)$, this means that $x_4 f = 0$; that is, if

$$x_4 f(x_1, x_2, x_3, x_4, x_5) \in (X_4^2, X_5^2 - X_1 X_3^2, X_3 X_4, X_4 X_5) \Rightarrow f \in Ann(x_4).$$

($x_3 \in Ann(x_4)$, because $x_3 x_4 = 0$;

$x_4 \in Ann(x_4)$, because $x_4 x_4 = 0$;

$x_5 \in Ann(x_4)$, because $x_5 x_4 = 0$.)

Ann (x_4) is an associated prime ideal in $\overline{T}$; that is a prime ideal which is the annihilator of some non-zero element of $\overline{T}$; that is Ann (x_4) is a prime ideal which is the annihilator of the non-zero element x_4 in $\overline{T}$. Then we have

$$depth\overline{T} = 1,$$

(since for the $R-$sequence $\{x_1, x_3, x_4, x_5\}$, only x_1 is not a zero divisor in $\overline{T}$.) And $dim\overline{T} = 2$. So we get

$$depth\overline{T} \neq dim\overline{T},$$

and hence $\overline{T}$ is not a Cohen-Macaulay ring. Since

$$\overline{T} = T/Tx_2$$

is not Cohen-Macaulay, T cannot be a Cohen-Macaulay ring.

In our example, we want to show that the coordinate ring $k[V]$ is not a Cohen-Macaulay ring. We have $char(k) = 2$, let's look at

$$f_1 = X_5^2 - X_1 X_3^2,$$

$$f_2 = X_4^2 - X_1X_2^2$$

in $R = k[X_1, X_2, X_3, X_4, X_5]$, f_1 and f_2 are relatively prime, so

$$V = V(f_1, f_2)$$

is a complete intersection in $\mathbb{A}^5$. Since $R = k[X_1, X_2, X_3, X_4, X_5]$ is a Cohen-Macaulay ring and (f_1, f_2) is a complete intersection in R, then by Proposition 4.7, we have

$$\overline{R} = R/(f_1, f_2)$$

is a Cohen-Macaulay ring.

$$x_5^2 = x_1x_3^2,$$

$$x_4^2 = x_1x_2^2$$

in $\overline{R} = R/(f_1, f_2) = k[X_1, X_2, X_3, X_4, X_5]/(X_5^2 - X_1X_3^2, X_4^2 - X_1X_2^2)$ it follows that

$$(x_4x_5)^2 = (x_1x_2x_3)^2,$$

and

$$(x_5x_2)^2 = (x_3x_4)^2.$$

Since $char(k)$ is 2, the elements

$$x_4x_5 - x_1x_2x_3,$$

and

$$x_5x_2 - x_3x_4$$

are nilpotent in R. Therefore we have

$$k[V] = k[X_1, X_2, X_3, X_4, X_5]/I(V) = R/I(V) \cong T.$$

As we have shown above, T is not a Cohen-Macaulay ring, so we have that the coordinate ring $k[V]$ is not a Cohen-Macaulay ring also.

Example 5.6 This is another example of an irreducible algebraic set which is not a complete intersection. (This is the more explicit form of Example 5.2 above). Let

$$V = \{(t, tu, u(u-1), u^2(u-1)) \in \mathbb{A}^4 \quad : \quad u, t \in k\}$$

be the image of the (u, t)-plane in the 4-dimensional space. Let

$$f_1 = X_1X_4 - X_2X_3 = 0,$$

$$f_2 = X_1^2X_3 + X_1X_2 - X_2^2 = 0,$$

$$f_3 = X_3^3 + X_3X_4 - X_4^2 = 0.$$

It is easy to see that

$$V = V(f_1, f_2, f_3)$$

and

$$I(V) = (f_1, f_2, f_3).$$

Now the morphism from the (u, t)-plane $\mathbb{A}^2$ onto $V \subset \mathbb{A}^4$ is obviously bijective with one exception: $(0, 0)$ and $(1, 0)$ in $\mathbb{A}^2$ are both mapped onto the origin $P = (0, 0, 0, 0) \in V$.

Since V is irreducible, $V - \{P\}$ is locally connected at P, but this does not hold formally.

More explicitly:

If we look at the completion $R^* = k[[X_1, X_2, X_3, X_4]]$ of the local ring $k[X_1, X_2, X_3, X_4]$, we see that from $(t, tu, u(u-1), u^2(u-1))$,

$$X_3 = U^2 - U$$

is solvable, this means that we can write U as a power series of X_3 with integer coefficients. (For more details see Geyer [5], page 249).

Now we have in R^* the splittings

$$f_2 = (UX_1 - X_2)(UX_1 - X_1 + X_2),$$

$$f_3 = (UX_3 - X_4)(UX_3 - X_3 + X_4).$$

Then we get

$$I^* = Q_1 \cap Q_2$$

where

$$Q_1 = (UX_1 - X_2, UX_3 - X_4),$$

$$Q_2 = ((U-1)X_1 - X_2, (U-1)X_3 - X_4),$$

so V splits at P into two surfaces which are complete intersections:

$$V^* = Spec(R^*/I^*) = V(Q_1) \cup V(Q_2).$$

But if we look at the ideal generated by Q_1 and Q_2 together, we see that

$$Q_1 + Q_2 = (X_1, X_2, X_3, X_4).$$

This is the maximal ideal P^* of R^*, so

$$V(Q_1) \cap V(Q_2) = \{P^*\}.$$

Therefore $Spec(R^*/I^*) - \{P^*\}$ is not connected. So by Theorem 4.10, we say that V is not a set-theoretic complete intersection locally at P.

Chapter 6

Complete Intersections in Characteristic $p > 0$

In this chapter, we report on a paper of Hartshorne [7] called 'Complete Intersections in Characteristic $p > 0$' and we show that for an algebraically closed field k, certain rational curves of degree $d \geq 4$ in $\mathbb{P}^3$ are set-theoretical complete intersections in all prime characteristics $p > 0$.

Recall that we call a variety V of dimension r in projective space $\mathbb{P}^n$ *a set-theoretical complete intersection* if there are $n - r$ hypersurfaces whose intersection is V.

We list below some results about characteristic $p > 0$.

- Cowsik and Nori [2] have shown that every curve in affine n-space $\mathbb{A}^n$ over a field of characteristic $p > 0$ is a set-theoretical complete intersection.

- Ferrand [4] has found a class of curves, including those curves of degree ≥ 4 in $\mathbb{P}^3$ mentioned in this chapter, which are set-theoretical complete intersections in every prime characteristic.

- Moh [12] has shown that every monomial curve in $\mathbb{P}^n_k$ is a set-theoretical complete intersection, where k is a field of characteristic $p > 0$.

  And the other result given by Moh in his paper is: Let k be a field of characteristic p and C a curve of $\mathbb{P}^n_k$. If there is a linear projection $\Pi : \mathbb{P}^n_k \to \mathbb{P}^2_k$ with center of Π disjoint of C, $\Pi(C)$ is birational to C and $\Pi(C)$ has only cusps as singularities, then C is a set-theoretical complete intersection.

- Robbiano and Valla [13] have shown that arithmetically Cohen-Macaulay curves are set-theoretic complete intersections in any characteristic.

  To define "arithmetically Cohen-Macaulay curves" let C be a monomial space curve in $\mathbb{P}^3$ given parametrically by

$$\begin{aligned} x &= u^m \\ y &= u^n t^{m-n} \\ z &= u^p t^{m-p} \\ w &= t^m, \end{aligned}$$

where m, n, p are positive integers such that $m > n$ and $m > p$. The curve C is called *arithmetically Cohen-Macaulay* iff the minimal number of generators of the defining ideal $I(C)$ is less than or equal to 3.

For example, the curve (t^4, t^6, t^7, t^9) is a set-theoretical complete intersection in $\mathbb{A}^4$. This curve is 'associated' to the curve $(u^5, u^3v^2, u^2v^3, v^5)$, and the binomial $(u^9 - v^4)$ in the sense of Thoma (see [14], example 2.2). Here the latter curve $(u^5, u^3v^2, u^2v^3, v^5)$ is arithmetically Cohen-Macaulay and is therefore a set-theoretic complete intersection of

$$X_2^2 - X_1X_3^2$$

and

$$X_3^5 - 3X_2^2X_3^2X_4 + 3X_1X_2X_3X_4^2 - X_1^2X_4^3,$$

(see [14]).

Another example of an arithmetically Cohen-Macaulay curve is the curve (u^7, u^6v, u^4v^3, v^7) in $\mathbb{P}^3$. This curve is a set-theoretic complete intersection of

$$X_2^3 - X_1^2X_3$$

and

$$X_3^7 - 3X_2^2X_3^4X_4 + 3X_1^2X_2X_3^2X_4^2 - X_1^4X_4^3$$

(see [14], example 3.5).

First let us define the curves which we will study:

Definition 6.1 *Let d be a positive integer, and let C_d be the curve in $\mathbb{P}_k^3$ given parametrically by*

$$x = tu^{d-1}$$

$$\begin{aligned} y &= t^{d-1}u \\ z &= t^d \\ w &= u^d, \end{aligned}$$

where u and t are in k, and $(t,u) \neq (0,0)$. This curve is a nonsingular rational curve of degree d.

Let us show the nonsingularity of the curve C_d:

To see that this curve is nonsingular, we should look at the Jacobian :

$$\begin{pmatrix} \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} & \frac{\partial w}{\partial t} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} & \frac{\partial w}{\partial u} \end{pmatrix} = \begin{pmatrix} u^{d-1} & (d-1)ut^{d-2} & dt^{d-1} & 0 \\ (d-1)tu^{d-2} & t^{d-1} & 0 & du^{d-1} \end{pmatrix}$$

For $t \neq 0$ and $u \neq 0$, we have that the rank of the above matrix is 2, so the curve C_d is nonsingular.

For example, C_4 is a set-theoretical complete intersection in characteristic $p = 3$ in $\mathbb{P}^3_k$. In fact,

$$C_4 = V(f_1, f_2)$$

where

$$f_1 = y^{13} - x^3 z^9 w$$

and

$$f_2 = z^9 w^{27} + \sum_{k=1}^{13} (-1)^k \binom{13}{k} y^{13-k} x^{36-3(13-k)} w^{2(13-k)}.$$

We find these surfaces by using the algorithm given in the proof of the following theorem:

Theorem 6.2 *Hartshorne [7]. Let k be an algebraically closed field of characteristic $p > 0$, then C_d is a set-theoretical complete intersection in $\mathbb{P}^3_k$ for any $d \geq 4$.*

Proof : The degree $d \geq 4$ of the curve, and the characteristic $p > 0$ of the field are given. We will choose integers $n > 0$ and r such that, if $q = p^n$, then the following two inequalities are satisfied :

$$(1 + \frac{1}{(d-1)})q < r < (1 + \frac{1}{(d-2)})q$$

and

$$r \leq (1 + \frac{1}{(d-1)})q + 1.$$

This is possible, by choosing n large enough so that

$$(1 + \frac{1}{(d-1)})q + 1 < (1 + \frac{1}{(d-2)})q$$

for example, and then taking r to be the largest integer $\leq (1 + \frac{1}{(d-1)})\ q + 1$.

From these inequalities, we can write the following:

$$r(d-1) - dq > 0$$

$$q(d-1) - r(d-2) > 0$$

$$r(d-1) - qd \leq d - 1$$

Now, we will consider the two surfaces with the following equations;

$$y^r = x^{r(d-1)-dq} z^q w^{q(d-1)-r(d-2)}$$

and

$$z^q w^{q(d-1)} + \sum_{k=1}^{r} (-1)^k \binom{r}{k} y^{r-k} x^{dq-(r-k)(d-1)} w^{(r-k)(d-2)} = 0.$$

We know that all the exponents of the above equations are non-negative by the choice of q and r.

Our aim is now to show that C_d can be written as the intersection of these $2 = codim(C_d, \mathbb{P}^3_k)$ surfaces.

If $w = 0$, then from the second equation we find $x = 0$, and from the first equation we find $y = 0$. So we say that there is only one point with $w = 0$, and this point is on the curve C_d.

Similarly, if $z = 0$, then from the first equation we find $y = 0$, and from the second equation we find $x = 0$. So there is only that one point, which is on C_d. Hence, to show that C_d is the intersection of these two surfaces, we can take $w = 1$, $x = t$, and assume $t \neq 0$. Now, it will be sufficient to show that the only common solution of those two equations is

$$y = t^{d-1}$$

and

$$z = t^d.$$

Let us take the second equation. After substituting $w = 1$ and $x = t$, we get

$$z^q - r y^{r-1} t^{dq-(r-1)(d-1)} + \ldots + (-1)^k \binom{r}{k} y^{r-k} t^{dq-(r-k)(d-1)} + \ldots + (-1)^r t^{dq} = 0.$$

Now, if we multiply the above equation by

$$t^{r(d-1)-dq},$$

we get

$$z^q t^{r(d-1)-dq} - ry^{r-1}t^{(d-1)} + \ldots + (-1)^k \binom{r}{k} y^{r-k}t^{k(d-1)} + \ldots + (-1)^r t^{r(d-1)} = 0.$$

From the first equation, by making these substitutions, we get

$$y^r = t^{r(d-1)-dq}z^q.$$

And we see that this is the first term of the above equation. So it becomes

$$y^r - ry^{r-1}t^{d-1} + \ldots + (-1)^k \binom{r}{k} y^{r-k}t^{k(d-1)} + \ldots + (-1)^r t^{r(d-1)} = 0.$$

It can be easily seen that, this is the expansion of the below expression

$$(y - t^{d-1})^r = 0.$$

Hence, we have that

$$y = t^{d-1}.$$

Now, from the first equation

$$y^r = t^{r(d-1)-dq}z^q$$

and from the above conclusion we find

$$y = t^{d-1},$$

we obtain

$$t^{r(d-1)} = t^{r(d-1)-dq}z^q.$$

Since $t \neq 0$ from our assumption, this implies

$$z^q - t^{dq} = 0.$$

And since $q = p^n$ is a power of the characteristic p, this implies

$$(z - t^d)^q = 0.$$

Hence, we conclude that $z = t^d$. $\square$

Remark 6.3 *Here for $d = 3$, we see that this curve is the twisted cubic curve, which is a set-theoretical complete intersection in any characteristic with the equations, for example $z^2 = yw$ and $y^3 - 2xyz + x^2w = 0$, as we have seen before (in page 25-26).*

By applying the procedure given in the proof we can show that C_4 is a set-theoretical complete intersection in characteristic $p = 2$ with the surfaces

$$y^{11} = xz^8w^2$$

and

$$z^8w^{24} + \sum_{k=1}^{11}(-1)^k \binom{11}{k} y^{11-k}x^{32-3(11-k)}w^{2(11-k)} = 0.$$

And, C_5 is a set-theoretical complete intersection in characteristic $p = 2$ with the surfaces

$$y^{21} = x^4z^{16}w$$

and

$$z^{16}w^{64} + \sum_{k=1}^{21}(-1)^k \binom{21}{k} y^{21-k}x^{80-4(21-k)}w^{3(21-k)} = 0.$$

To conclude this chapter, let us mention that the general problem of whether all projective curves are set-theoretical complete intersections in characteristic $p > 0$ is an open problem. The problem of whether every connected projective curve in $\mathbb{P}^3$ is a set-theoretical complete intersection is even not solved.

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