CURVES IN PROJECTIVE SPACE

A THESIS
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE INSTITUTE OF ENGINEERING AND SCIENCE
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Ali Yıldız
July, 2003

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Ali Sinan Sertöz (Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Alexander Klyachko

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Tuğrul Hakioğlu

Approved for the Institute of Engineering and Science:

Prof. Dr. Mehmet B. Baray
Director of the Institute Engineering and Science

ABSTRACT

CURVES IN PROJECTIVE SPACE

Ali Yıldız
M.S. in Mathematics
Supervisor: Assoc. Prof. Dr. Ali Sinan Sertöz
July, 2003

This thesis is mainly concerned with classification of nonsingular projective space curves with an emphasis on the degree-genus pairs. In the first chapter, we present basic notions together with a very general notion of an *abstract non-singular curve* associated with a function field, which is necessary to understand the problem clearly. Based on Nagata's work [25], [26], [27], we show that every nonsingular abstract curve can be embedded in some $\mathbb{P}^N$ and projected to $\mathbb{P}^3$ so that the resulting image is birational to the curve in $\mathbb{P}^N$ and still nonsingular. As genus is a birational invariant, despite the fact that degree depends on the projective embedding of a curve, curves in $\mathbb{P}^3$ give the most general setting for classification of possible degree-genus pairs.

The first notable attempt to classify nonsingular space curves is given in the works of Halphen [11], and Noether [28]. Trying to find valid bounds for the genus of such a curve depending upon its degree, Halphen stated a correct result for these bounds with a wrong claim of construction of such curves with prescribed degree-genus pairs *on a cubic surface*. The fault in the existence statement of Halphen's work was corrected later by the works of Gruson, Peskine [9], [10], and Mori [21], which proved the existence of such curves on quartic surfaces. In Chapter 2, we present how the fault appearing in Halphen's work has been corrected along the lines of Gruson, Peskine, and Mori's work in addition to some trivial cases such as genus 0, 1, and 2 together with hyperelliptic, and canonical curves.

Keywords: Abstract curve, nonsingular curve, hyperelliptic curve, discrete valuation ring, projective curve, projective embedding, genus, degree, degree-genus pair, quadric surface, cubic surface, quartic surface, quadric surface, moduli space.

ÖZET

PROJEKTİF UZAYDA EĞRİLER

Ali Yıldız
Matematik, Yüksek Lisans
Tez Yöneticisi: Doç. Dr. Ali Sinan Sertöz
Temmuz, 2003

Bu tez, esas olarak derece-cins çiftlerine odaklanarak, tekil olmayan projektif uzay eğrilerinin sınıflandırılması hakkındadır. Birinci bölümde, problemi açık biçimde anlamak için gerekli olan temel kavramları verilen bir fonksiyon cismine karşılık gelen tekil olmayan soyut eğri kavramının genel tanımıyla birlikte sunuyoruz. Nagata'nın çalışmalarından hareketle, tekil olmayan her soyut eğrinin bir $\mathbb{P}^N$'e gömülebileceğini ve oluşacak görüntü hala tekil olmayacak ve $\mathbb{P}^N$'deki eğriye birasyonel olacak biçimde $\mathbb{P}^3$'e izdüşürülebileceğini gösteriyoruz. Her ne kadar derece eğrinin projektif gömevine bağlı olsa da, cins birasyonel bir değişmez olduğundan $\mathbb{P}^3$'deki eğriler olası derece-cins çiftlerininin sınıflandırılması için en genel ortamı sağlamaktadır.

Tekil olmayan uzay eğrilerinin sınıflandırılması ile ilgili ilk kayda değer girişim Halphen [11] ve Noether'in [28] çalışmalarında görülmektedir. Dereceye bağlı olarak olası cins için geçerli bir aralık bulmaya çalşırken, Halphen bu aralık için doğru bir sonucu, bu derece ve cinse sahip tekil olmayan eğrileri *kübik bir yüzey* üzerinde kurduğu biçiminde yanlış bir iddia ile beraber belirtmiştir. Halphen'in çalışmasında görülen bu hata, daha sonradan Gruson, Peskine [9], [10] ve Mori'nin [21] çalışmaları ile ilgili eğrilerin dörtlenik yüzeyler üstündeki varlığı gösterilerek düzeltilmiştir. İkinci bölümde, hipereliptik eğrilerle beraber cinsin 0, 1 ve 2 olduğu bazı nisbeten kolay durumların incelenmesine ek olarak Halphen'in çalışmasında görülen yanlışın Gruson, Peskine ve Mori'nin çalışmaları ile nasıl düzeltildiğini gösteriyoruz.

Anahtar sözcükler: Soyut eğri, tekil olmayan eğri, hipereliptik eğri, ayrık valüasyon halkası, projektif eğri, projektif gömev, cins, derece, derece-cins çifti, ikilenik yüzey, kübik yüzey, dörtlenik yüzey, moduli uzayı.

Acknowledgement

I would like to express my thanks to my supervisor Ali Sinan Sertöz, firstly for introducing me to the field of algebraic geometry, and for all his patience, advice, and valuable guidance thereafter.

My thanks are also due to Alexander Klyachko, and Tuğrul Hakioğlu who accepted to review this thesis, and participated in the exam committee.

I would like to thank especially to the Institute of Engineering and Science of Bilkent University, and Directorate of Human Resources Development of TÜBİTAK which have financially supported the work which led to this thesis.

I would like to express my deepest gratitude to the candle of my life, Selen Gürkan, whose love always shined over my pathway through hard times, guiding my spirit towards the light.

I would like to thank Mustafa Keşir, Süleyman Tek, and Ergün Yaraneri for their valuable remarks, and suggestions in regard to typesetting problems in LaTeX.

Contents

Chapter 1

Introduction

1.1 Motivation and Historical Background

The principal theme of this thesis is the classification of nonsingular algebraic curves, sitting in $\mathbb{P}^3$, up to *birational* equivalence; by concentrating on the *degree-genus* pairs. On its own merit, classification problem has motivated much of the research in algebraic geometry. For the most part, problems concerning classification of algebraic varieties are hopelessly difficult to answer in general, but progress can be measured against them within appropriate restrictions. Some of the classical classification problems can be listed as follows;

- Classify all varieties up to isomorphism.
- Classify all nonsingular projective varieties up to birational equivalence. By a famous theorem of Hironaka every quasi-projective variety is birationally equivalent to a projective variety, and every projective variety is birationally equivalent to a nonsingular projective variety. Hence such a classification automatically leads to a birational classification of all quasi-projective varieties.
- Classify the varieties in each birational equivalence class up to isomorphism.

- Choose a canonical representative for each birational equivalence class.

For curves (one-dimensional varieties) all of these questions have satisfactory answers, which have been developed during centuries of beatiful mathematics.

Every rational map between curves extends uniformly to a well-defined morphism; hence birational maps and isomorphisms are the same for curves. It is relatively easy to prove that each birational class has a unique nonsingular projective model (cf. [13], page 45). Because complex curves are Riemann surfaces, classifying complex curves has led to an algebraic analogue of *Teichmüller theory*, which studies the moduli of Riemann surfaces up to conformal isomorphism. From the viewpoint of algebraic geometry the main results can be summarized as follows

- There exists only one genus-zero curve up to isomorphism, namely $\mathbb{P}^1$.
- There exists a one-parameter family of isomorphism classes of curves of genus one, the so-called *elliptic curves*, indexed by the $j-$invariant, a parameter varying over $\mathbb{A}^1$ (cf. [13] Chapter IV, Section 4).
- The curves of genus greater than one are parametrized by the *moduli spaces* $\mathcal{M}_g$. These moduli spaces were first constructed by David Mumford as abstract $(3g-3)$-dimensional varieties (cf. [22]) but soon afterward were shown to be, in fact, irreducible quasi-projective varieties by Deligne and Mumford (cf. [2]). The structure of these moduli spaces and their generalizations is an active field of research, especially since interesting questions with theoretical physics were discovered in the past ten years by Witten, Kontsevich, and others; cf. [12].

As the above summary indicates, quite a lot of information about the classification of curves is known. Nevertheless, questions still abound. For example, although every nonsingular projective curve can be embedded into projective three-space $\mathbb{P}^3$, it is still unknown whether or not every such curve is the intersection of two surfaces.

To appreciate the setting of the problem better, a very general notion of an "abstract nonsingular projective curve" will be presented on the foregoing pages, and we will prove that every nonsingular projective variety of dimension r can be embedded in $\mathbb{P}^{2r+1}$. In particular for nonsingular projective curves with dimension 1, every such curve can be embedded into $\mathbb{P}^3$. A natural question arising within the classification problem is to ask what kind of pairs $(d, g) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ can occur such that a nonsingular algebraic curve $\mathcal{C} \subseteq \mathbb{P}^3$ with degree d and genus g exists. Classification for curves with low genus, namely 0,1, and 2, are rather easy to deal with, and we will do this. For curves with higher genus, a fruitful approach has been gained by asking the least degree of a surface in $\mathbb{P}^3$ on which curve lies.

The question regarding the existence of degree-genus pairs $(d, g) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ were already studied extensively in the late 19th century, especially in the works of Halphen [11] and Noether [28], who shared the Steiner Prize in 1882. In his research, Halphen presented a theorem which gives an upper bound for the genus g of nonsingular algebraic curves in $\mathbb{P}^3$ depending upon the degree d of the curve, provided that the curve does not lie on either a linear subspace, or a quadric surface in $\mathbb{P}^3$. In his research, Halphen claimed to construct each such curve with a prescribed degree $d > 0$ and genus g with $0 \leq g \leq \frac{1}{6}d(d-3)+1$ on cubic surfaces. However, later it has been shown that construction of some curves with genus lying in the asserted interval is not possible even on a singular cubic surface. Solution of this problem was completed in $1982 - 1984$ by the works of Gruson, Peskin, and Mori proving the existence of curves with genus g within the asserted interval on a nonsingular quartic surface in $\mathbb{P}^3$.

1.2 Preliminaries: Basic Definitions

In order to fill the reader's wonder why we are interested in curves in $\mathbb{P}^3$, and also to set up the problem on strongly built mathematical basis, we start by giving the basic definitions pertaining to the concepts that will show up in the narration of the problem. These explanations will also make it clear why we are interested in curves in $\mathbb{P}^3$, and why it generalizes naturally to the classification of all nonsingular projective curves up to birational equivalence.

Before giving the definitions of concepts which will show up in the depiction of the problem this writing is concerned, we a priori assume some familiarity with well-known basic concepts in commutative algebra, such as *"Noetherian rings"*. Simply defining it, a ring R in which every ascending chain of ideals terminates is called a Noetherian ring. By an ascending chain of ideals in a ring R, we mean the existence of ideals $I_1, I_2, \ldots, I_n, \ldots$ of R with the property that $I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq I_{n+1} \ldots$ And such a chain terminates if there is an index N such that $\forall n \geq N, \qquad I_n = I_N$. In this case, all ascending chains of ideals indeed will contain only finitely many proper ideal inclusions in ordering by inclusion. To be able to make use of Hilbert's Nullstellensatz and other advantages of commutative algebra, from here on we conveniently assume that our ground field k is an algebraically closed field, i.e. $k = \overline{k}$.

The affine n-space $\mathbb{A}_k^n$ (denoted $\mathbb{A}^n$ when the field of discourse is clear from the context) is defined as

$$\mathbb{A}_k^n \ = \ \{(a_1, \ldots, a_n) \mid a_i \in k \qquad \forall\ i = 1, \ldots, n\}$$

which has the same underlying set as k^n but without a vector space structure. Given a subset T of $k[x_1, \ldots, x_n]$, we define its zero set as

$$Z(T) \ = \ \{P \in \mathbb{A}_k^n \mid f(P) = 0 \quad \forall\ f \in T\}$$

It is clear that $Z(T) \ = \ Z((T)_{ideal})$ where $(T)_{ideal}$ is the ideal in $k[x_1, \ldots, x_n]$ which is generated by the set T. A set $A \subseteq \mathbb{A}^n$ is called an algebraic set in case

$A = Z(T)$ for a subset $T \subseteq k[x_1, \ldots, x_n]$. Conversely, given a subset A of $\mathbb{A}^n$ we define its defining ideal $I(\mathrm{A})$ as

$$I(\mathrm{A}) \;=\; \{f \in k[x_1, \ldots, x_n] \mid \qquad f(P) = 0 \quad \forall\, P \in \mathrm{A}\}$$

An algebraic set whose defining ideal is a prime ideal is called a *variety*. Naturally, there arises two maps first of which mapping a given subset X of $\mathbb{A}^n$ to its ideal in $k[x_1, \ldots, x_n]$(i.e. $X \subseteq \mathbb{A}^n \to I(X)$), and the second of which mapping a given ideal of $k[x_1, \ldots, x_n]$ to its zero set in $\mathbb{A}^n$(i.e. $I \subseteq k[x_1, \ldots, x_n] \to Z(I)$). It is again an elementary fact that both maps are inclusion-reversing maps.

The affine n-space $\mathbb{A}^n$ can be topologized by defining the open sets to be the complements of algebraic sets, and the so-defined topology is called the Zariski topology. With respect to this topology, a quasi-variety is an open subset of a variety in $\mathbb{A}^n$. The following are the elementary properties regarding the zero sets and defining ideals: Let X_1, X_2 be subsets of $\mathbb{A}^n$ and T_1, T_2 be subsets of $k[x_1, \ldots, x_n]$. Then

- $T_1 \subseteq T_2 \;\Rightarrow\; Z(T_1) \supseteq Z(T_2)$
- $Y_1 \subseteq Y_2 \;\Rightarrow\; I(Y_1) \supseteq I(Y_2)$
- $I(Y_1 \cup Y_2) \;=\; I(Y_1) \cup I(Y_2)$.
- For any ideal $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$, $\quad I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$, i.e. the radical of $\mathfrak{a}$.
- For any subset $X \subseteq \mathbb{A}^n$, $\quad Z(I(X)) = \overline{X}$, the closure of X with respect to Zariski topology.

The statement $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$ appearing above is a direct consequence of Hilbert's Nullstellensatz, which is valid only on an algebraically closed field $k = \overline{k}$. To see this, as a very simple counter-example let $k = \mathbb{R}$ and consider the variety in $\mathbb{A}^2$ given by the equation $X : y = 0$ then since the polynomial $x^2 + 1$ has no zeroes in $\mathbb{R}$, $Z(y(x^2+1)) = Z(y) \subseteq \mathbb{A}^2$. Then if the result mentioned above still holds we must have $I(Z(y(x^2+1))) = I(Z(y)) \Leftrightarrow \sqrt{(y(x^2+1))} = \sqrt{(y)} \Rightarrow$

$x^2 + 1 \in (y) \Rightarrow x^2 + 1 = yp(x, y)$ for some $p(x, y) \in \mathbb{R}[x, y]$ which is a plain contradiction.

Projective n-space over a field k denoted by $\mathbb{P}^n_k$ (or by $\mathbb{P}^n$ when there is no confusion as to the field in consideration) is defined as the space consisting of equivalence classes of the equivalence relation $\sim$ defined on $\mathbb{A}^{n+1} \backslash \{(0, \ldots, 0)\}$, so formally;

$$\mathbb{P}^n_k = \frac{\mathbb{A}^{n+1} \backslash \{(0, \ldots, 0)\}}{\sim} = \{(a_0, \ldots, a_n) \mid a_i \in k, \text{ not all } \mathrm{a_i} = 0\} / \sim$$

where the relation $\sim$ on $\mathbb{A}^{n+1} \backslash \{(0, \ldots, 0)\}$ is defined, for given any two points $P = (a_0, \ldots, a_n)$, $Q = (b_0, \ldots, b_n) \in \mathbb{A}^{n+1} \backslash \{(0, \ldots, 0)\}$, as :

$$P \sim Q \iff \exists \lambda \in k^\times \quad \text{such that} \quad a_i = \lambda \cdot b_i \quad \forall i = 0, \ldots, n.$$

An algebraic set in $\mathbb{P}^n$, is defined to be the zero set of a set of **homogeneous** polynomials, i.e. $Z(T)$ where T is a set of homogeneous polynomials. A variety in $\mathbb{P}^n$ (or in other terms a projective variety) is defined as a projective algebraic set whose defining ideal in $k[x_0, \ldots, x_n]$ is a prime ideal. Homogeneity requirement in projective definitions is necessary to make the zero value of the polynomial independent of different representations of coordinates in $\mathbb{P}^n$, which are the equivalence classes obtained from $\mathbb{A}^{n+1}$. Zariski topology on $\mathbb{P}^n$ is defined similarly, where open sets are the complements of projective algebraic sets, and a quasi-projective variety is defined as an open subset of a projective variety.

This very definition of topology and the inclusion-reversed descending chain of closed sets of $\mathbb{A}^n$ in correspondence to the ascending chain of ideals in $k[x_1, \ldots, x_n]$ motivates the definition of a "*Noetherian* topological space". As easily predictable, a topological space X is called Noetherian if every descending chain of closed subsets of X terminates, i.e. if $Z_1, \ldots, Z_n, \ldots$ are closed subsets of X subject to the condition $Z_1 \supseteq Z_2 \supseteq \ldots \supseteq Z_n \supseteq Z_{n+1} \supseteq \ldots$, there is an index N such that $\forall \mathrm{n} \geq N, \quad Z_n = Z_N$.

A trivial example to a Noetherian topological space is the affine n-space $\mathbb{A}^n$ equipped with Zariski topology, since if $Z_1 \supseteq Z_2 \supseteq \ldots \supseteq Z_n \supseteq Z_{n+1} \supseteq \ldots$ is any descending chain of closed subsets of $\mathbb{A}^n$, then $I(Z_1) \subseteq I(Z_2) \subseteq \ldots \subseteq I(Z_n) \subseteq I(Zn+1) \subseteq \ldots$ is an ascending chain of ideals in $k[x_1, \ldots, x_n]$ which is a Noetherian ring by Hilbert's Basis Theorem, hence this chain of ideals must terminate. Therefore $\exists\ N \in \mathbb{N}$ such that $\forall\ n \geq N, \quad I(Z_n) = I(Z_N)$. Then going back to $\mathbb{A}^n$, $\forall\ n \geq N \quad Z(I(Z_n)) = Z(I(Z_N)) \Leftrightarrow \overline{Z_n} = \overline{Z_N} \Leftrightarrow \forall\ n \geq N, \quad Z_n = Z_N$.

1.3 Dimension

Recall that in a commutative ring R with unity 1, an ideal Q is called a primary ideal if, for any $a, b \in R$, $ab \in Q$, and $b \notin Q \Rightarrow\ \exists n \in \mathbb{N}$ such that $a^n \in Q$. Moreover it is a standard algebraic result that every ideal $\mathfrak{a}$ in a Noetherian ring R can be written as an intersection of finitely many primary ideals (whose radicals are prime ideals), i.e. any ideal $\mathfrak{a}$ of R can be written as $\mathfrak{a} = \mathfrak{I}_\mathfrak{1} \cap \ldots \cap \mathfrak{I}_\mathfrak{s}$ with $\mathfrak{I}_\mathfrak{i}$ is a primary ideal and $\sqrt{\mathfrak{I}_\mathfrak{i}} = \mathfrak{P}_\mathfrak{i}$ is prime for $i = 1, \ldots, s$, unique up to the exchange of places, no one containing any other, a result proved for $k[x_1, \ldots, x_n]$ first in 1905 by E.Lasker who was the world chess champion from 1894 to 1921 (cf. [1], pp. 338-344). A primary ideal $\mathfrak{I}$ in the ring $k[x_1, \ldots, x_n]$ has the property that $\sqrt{\mathfrak{I}} = \mathfrak{P}$ is a prime ideal, and hence $Z(\mathfrak{I}) = Z(\sqrt{\mathfrak{I}}) = Z(\mathfrak{P})$ by Hilbert's Nullstellensatz and hence a variety. Since the concept of the *Noetherian topological space* was motivated by the concept of the *Noetherian ring* , we might expect a similar **decomposition result** to hold in a Noetherian topological space. This result summarizes as follows :

Proposition 1.3.1 *Any subset Y of a Noetherian topological space X can be decomposed into irreducible closed subsets $Y_1, \ldots, Y_n$ of X, i.e. we can write $X = \bigcup_{i=0}^n Y_i$. If we impose the extra condition that for any i, j with $i \neq j$*

$Y_i \nsubseteq Y_j$, *the decomposition is unique as to the components* $Y_1, \ldots, Y_n$, *which are called the irreducible components of* Y.

Proof:
(Existence) Suppose to the contrary that there is a subset Y of the Noetherian topological space X, which does not have a decomposition into finitely many closed subsets of X. Since Y cannot be irreducible which would imply trivial decomposability, we can write $Y = Y_1 \cup Y_2$ where Y_1, Y_2 are proper closed subset of Y. At least one of Y_1 and Y_2 must be indecomposable, Without loss of generality suppose it is Y_1, Now at each step applying the same argument to Y_n as the one applied to Y, pick an indecomposable component Y_{n+1} of Y_n inductively. But since each Y_{n+1} is a proper closed subset of Y_n, We have $Y_{n+1} \subsetneqq Y_n$. Then we get a proper chain of descending closed sets $Y \supsetneqq Y_1 \supsetneqq \ldots Y_n \supsetneqq Y_{n+1} \supsetneqq \ldots$, which is a contradiction since the space X is Noetherian. Hence every subset must be decomposable into finitely many irreducible closed subsets of X.

(Uniqueness) Suppose that for any i, j with $i \neq j$, $Y_i \nsubseteq Y_j$, and let $Y_1 \cup \ldots \cup Y_n$ and $Z_1, \cup \ldots \cup, Z_m$ be two different decompositions of Y, subject to our extra condition. Then since $Z_1 \subseteq Y = Y_1 \cup \ldots \cup Y_n$, We are allowed to write $Z_1 = \bigcup_{i=1}^{n}(Y_i \cap Z_1)$, but Z_1 was irreducible by assumption. Thus components appearing in the union on the right hand side must not be proper, which means all are $\emptyset$ except one. Without loss of generality (by renumbering if necessary) supposing this nonempty term to be $Y_1 \cap Z_1$, we have $Z_1 = Y_1 \cap Z_1$, or equivalently $Z_1 \subseteq Y_1$. Applying the symmetric argument to Y_1 we have only one of $Y_1 \cap Z_k$, $k = 1, \ldots, n$ is non-empty and it can be only $Y_1 \cap Z_1$, hence $Y_1 \subseteq Z_1$. So $Z_1 = Y_1$. By applying induction on n, We get the equality of all irreducible components, thereby proving the proposition in full. □

So, by this proposition every algebraic set can be decomposed into finitely many varieties, and moreover by considering the relative Zariski topology we can talk about the decomposition of quasi-affine, quasi-projective varieties. This result motivates the following definition.

Definition 1.3.1 *Given a Noetherian topological space X, its dimension, denoted* $\dim X$*, is defined as*

$$\dim X = \mathrm{Sup}\{n \in \mathbb{N} \mid \exists\, \emptyset \neq X_0, \ldots, X_n = X \ \textit{such that}\ X_0 \subsetneqq X_1 \ldots \subsetneqq X_n\}$$

where the sets $X_0, \ldots, X_n$ are closed irreducible subspaces of X.

By the preceding proposition, this definition makes sense since we can have only finitely many closed subspaces of X in every such chain. This definition of dimension easily extends to the projective and quasi cases by considering relative topologies. To give an example, $\mathbb{A}^1$ has dimension 1, since its only irreducible closed subspaces are single-point sets and the whole space $\mathbb{A}^1$.

Varieties of dimension 1,2,..., n are called curves, surfaces, ..., n-folds. Our interest throughout this thesis will be focused on curves, and surfaces, precisely "curves on surfaces".

1.4 Ring of Regular Functions

At this stage, we need to make it clear what we mean by an isomorphism and birational equivalence between two varieties. For this purpose we will give a brief review of these two concepts.

Definition 1.4.1 *Let Y be an affine or quasi-affine variety, a function $f : Y \to k$ is called a regular function at a point P of Y, if f can be represented as*

$$f = \frac{g}{h}$$

on an open set U containing P, where $g, h \in k[x_1, \ldots, x_n]$, and $h \neq 0$ on U. In case f is regular at every point of Y, it is called regular on Y.

Observe that since the variety Y is an irreducible algebraic set, any two open subset U, V of Y has to intersect. As otherwise, $U \cap V = \emptyset \Rightarrow Y = (Y - U) \cup (Y - V)$ which makes Y reducible, and hence contradiction. So we can define the addition and multiplication of different elements $(U, f), (V, g)$ (f being regular on U, and g regular on V) of the set of regular functions on the intersection of where they are defined, namely U and V, making the set of regular functions on Y into a ring, denoted $\mathcal{O}_Y$. In order for this definition to make sense in the projective and quasi-projective cases, and hence $f = \frac{g}{h}$ to be well-defined, independent of different representatives of homogeneous coordinates, we must require g, and h be homogeneous polynomials and $deg(g) = deg(h)$. So the projective definition follows:

Definition 1.4.2 *Let Y be a projective or quasi-projective variety, a function $f : Y \to k$ is called a regular function at a point P of Y, if f can be written as*

$$f = \frac{g}{h}$$

on an open set U containing P, where $g, h \in k[x_0, \ldots, x_n]$ are homogeneous polynomials satisfying $deg(g) = deg(h)$, and $h \neq 0$ on U. In case f is regular at every point of Y, it is called regular on Y.

By a $\mathcal{C}^\infty$ map $f : \mathbb{R}^m \to \mathbb{R}^n$ we understand usually a map $f = (f^1, \ldots, f^n)$ such that whose components, i.e. each of f^i, is differentiable of any requested order with respect to each of its arguments $x_1, \ldots, x_m$. A morphism between two $\mathcal{C}^\infty$ manifolds M and N is defined as a map $\psi : \mathrm{M} \to \mathrm{N}$, which is a $\mathcal{C}^\infty$ mapping. Smoothness is defined locally, for any open set $\mathrm{U} \subseteq \mathrm{M}$, and $\psi(\mathrm{U}) \subseteq \mathrm{V} \subseteq \mathrm{N}$ with charts $\phi_{\mathrm{U}} : \mathrm{U} \to \mathbb{R}^m$ and $\varphi_V : \mathrm{V} \to \mathbb{R}^n$, we must have

$$\varphi_{\mathrm{V}} \circ \psi \circ \psi_{\mathrm{U}}^{-1} : \mathbb{R}^m \to \mathbb{R}^n \qquad \text{is} \quad \mathcal{C}^\infty \quad \text{in the usual sense.}$$

In differential geometry, a $\mathcal{C}^\infty$ manifold M is always mentioned with its differential structure $\mathcal{C}^\infty(M)$ which is defined as

$$\mathcal{C}^\infty(M) = \{f : M \to \mathbb{R} \ | \ f \ is \ a \ \mathcal{C}^\infty \ function\}$$

which has a natural ring structure where addition and multiplication are defined pointwise. The functions in $\mathcal{C}^\infty(M)$ are called the smooth maps on the manifold M. These maps are considered in direct analogy to what is defined as regular maps in algebraic geometry. Let ψ : M $\to$ N be a set-theoretic map. If f : V $\subseteq$ N $\to \mathbb{R}$ is an $\mathbb{R}$-valued function on an open set V $\subseteq$ N, the composition $f \circ \psi : \psi^{-1}(\mathrm{U}) \to \mathbb{R}$ is again a set-theoretic function. It is denoted by $\psi^* f$ and is called the *pull-back* of f by ψ. Indeed this way a differential structure on a manifold N can be carried directly to another manifold M, i.e. pulling back the differential structure $\mathcal{C}^\infty(N)$ to over M. The following proposition shows the exact motivation for the definition of morphism in algebraic geometry.

Proposition 1.4.1 *Let M and N be $\mathcal{C}^\infty$ manifolds of real-dimension m and n. A function $\psi : M \to N$ is a morphism (i.e. a $\mathcal{C}^\infty$ map) $\iff$ for every $\mathcal{C}^\infty$ map $f : V \subseteq N \to \mathbb{R}$, the function $f \circ \psi : \psi^{-1}(V) \subseteq M \to \mathbb{R}$ is a $\mathcal{C}^\infty$ map. (i.e. $\psi : M \to N$ is a morphism $\iff \psi^*(\mathcal{C}^\infty(V)) \subseteq C^\infty(\psi^{-1}(V))$ for any open subset V of N.)*

Proof:

$\Rightarrow$:

Suppose that ψ : M $\to$ N is a morphism, i.e. $\mathcal{C}^\infty$ map. Then locally, for any arbitrary open set U $\subseteq$ M, and ψ(U) $\subseteq$ V $\subseteq$ N (V open in N) with charts $\phi_{\mathrm{U}} : \mathrm{U} \to \mathbb{R}^m$ and $\varphi_V : \mathrm{V} \to \mathbb{R}^n$, we have $\varphi_{\mathrm{V}} \circ \psi \circ \phi_{\mathrm{U}}^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ *is* $\mathcal{C}^\infty$ in the usual sense. Then pick a $\mathcal{C}^\infty$ map f : V $\subseteq$ N $\to \mathbb{R}$. f is $\mathcal{C}^\infty$ means that the map

$$f \circ \varphi_{\mathrm{V}}^{-1} : \varphi_V(\mathrm{V}) \subset \mathbb{R}^n \to \mathbb{R} \qquad \text{is a} \quad \mathcal{C}^\infty \quad \text{map in the usual sense.}$$

Now consider the function $f \circ \psi : \psi^{-1}(\mathrm{V}) \subseteq \mathrm{M} \to \mathbb{R}$. Writing $f \circ \psi$ in local coordinates, for any open set U $\subseteq$ M with $\mathrm{U} \cap \psi^{-1}(\mathrm{V}) \neq \emptyset$ with chart $\phi_{\mathrm{U}} : \mathrm{U} \to$

$\mathbb{R}^m$, in order for $f \circ \psi : \psi^{-1}(\mathrm{V}) \subseteq \mathrm{M} \to \mathbb{R}$ to be a $\mathcal{C}^\infty$ map, the map

$$f \circ \psi \circ \phi_{\mathrm{U}}^{-1} : \phi_{\mathrm{U}}(\mathrm{U}) \subseteq \mathbb{R}^m \to \mathbb{R} \qquad \text{must be a } \mathcal{C}^\infty \text{ map in the usual sense.}$$

But, observe that $f \circ \psi \circ \phi_{\mathrm{U}}^{-1} = \underbrace{(f \circ \varphi_{\mathrm{V}}^{-1})}_{\in\ \mathcal{C}^\infty(\mathbb{R}^n)} \circ \underbrace{(\varphi_{\mathrm{V}} \circ \psi \circ \phi_{\mathrm{U}}^{-1})}_{\in\ \mathcal{C}^\infty(\mathbb{R}^m \to \mathbb{R}^n)}$ composition of two $\mathcal{C}^\infty$ maps in the usual sense (with respective domains and ranges overlapping perfectly). Therefore, by virtue of the so-called "`Chain Rule`" the function $f \circ \psi \circ \phi_{\mathrm{U}}^{-1}$ is a $\mathcal{C}^\infty$ function in the usual (Euclidean) sense.

To state the conclusion, $\psi : \mathrm{M} \to \mathrm{N}$ is a morphism (i.e. a $\mathcal{C}^\infty$ map) $\Rightarrow$ for any $\mathcal{C}^\infty$ map $f : \mathrm{V} \subseteq \mathrm{N} \to \mathbb{R}$, the function $f \circ \psi : \psi^{-1}(\mathrm{V}) \subseteq \mathrm{M} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ map.

$\Leftarrow$:

Clearly the i-th projection map $\pi_i : \mathbb{R}^n \to \mathbb{R}$ which maps any given point $(x_1, \ldots, x_n)$ to its i-th coordinate x_i is a $\mathcal{C}^\infty$ map in the usual sense. Therefore, for any $i = 1, \ldots, n$ the function $f_i : \mathrm{N} \to \mathbb{R}$ defined locally on any open set $\mathrm{V} \subseteq N$ with chart $\varphi_{\mathrm{V}} : \mathrm{V} \to \mathbb{R}^n$ in the way $f_i \mid_{\mathrm{V}} = \pi_i \circ \varphi_{\mathrm{V}}$ is a $\mathcal{C}^\infty$ map, as $f_i \circ \varphi_{\mathrm{V}}^{-1} = (\pi_i \circ \varphi_{\mathrm{V}}) \circ \varphi_{\mathrm{V}}^{-1} = \pi_i : \varphi_{\mathrm{V}}(\mathrm{V}) \subseteq \mathbb{R}^n \to \mathbb{R}$ is $\mathcal{C}^\infty$. Now by the hypothesis, $f_i \circ \psi : \psi^{-1}(\mathrm{V}) \subseteq \mathrm{M} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ map. But $f_i \circ \psi \in \mathcal{C}^\infty(\mathrm{U} \cap \psi^{-1}(\mathrm{V})) \iff \pi_i \circ \varphi_{\mathrm{V}} \circ \psi \circ \phi_{\mathrm{U}}^{-1} : \phi_{\mathrm{U}}(\mathrm{U}) \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is a $\mathcal{C}^\infty$ function in the usual sense. But then each component function of $\varphi_{\mathrm{V}} \circ \psi \circ \phi_{\mathrm{U}}^{-1} : \phi_{\mathrm{U}}(\mathrm{U}) \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is a $\mathcal{C}^\infty$ map, whence $\varphi_{\mathrm{V}} \circ \psi \circ \phi_{\mathrm{U}}^{-1} : \phi_{\mathrm{U}}(\mathrm{U}) \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is a $\mathcal{C}^\infty$ map.

To conclude, for the function $\psi : \mathrm{M} \to \mathrm{N}$; given any $\mathcal{C}^\infty$ map $f : \mathrm{V} \subseteq \mathrm{N} \to \mathbb{R}$, the function $f \circ \psi : \psi^{-1}(\mathrm{V}) \subseteq \mathrm{M} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ map $\Rightarrow \psi : \mathrm{M} \to \mathrm{N}$ is a morphism.

Summarizing the result:

$\therefore \quad \psi : \mathrm{M} \to \mathrm{N}$ is a morphism $\iff \psi^*(\mathcal{C}^\infty(V)) \subseteq \mathcal{C}^\infty(\psi^{-1}(V))$ for any open subset V of N. $\square$

This definition of a morphism easily extends to the case of $\mathcal{C}^r$, $\mathcal{C}^\omega$ (analytic) manifolds, Riemann surfaces and also to other classes of manifolds, such as the holomorphic ones. It motivates the following definition of 'morphism' in algebro-geometric setting.

Definition 1.4.3 *Let X and Y be two varieties, a continuous map $\phi : X \to Y$ is called a* ***morphism*** *if it pulls back regular functions to regular functions, i.e. if $\phi^*\mathcal{O}_Y(\mathrm{V}) \subseteq \mathcal{O}_X(\phi^{-1}(\mathrm{V}))$ for all open subsets $\mathrm{V} \subseteq \mathrm{Y}$. (i.e. for any regular function $f : \mathrm{V} \to k$ defined on any open set $\mathrm{V} \subseteq Y$, the function $f \circ \phi : \phi^{-1}(\mathrm{V}) \subseteq X \to k$ is also a regular function.)*

A morphism $\psi : X \to Y$ with a dense image $\psi(X)$ in its target variety Y is called a *dominant morphism.* In case ψ is a dominant morphism its image contains a non-empty subset (a quasi-projective variety) of Y. And as quite expected, an isomorphism is a morphism with a morphism inverse.

Consider pairs (U, f) where U is an open subset of X and $f \in \mathcal{O}_X(U)$ a regular function on X. Call two such pairs (U, f) and (U', f') equivalent, denoting $(U, f) \sim (U', f')$, if there is an open subset V in X with $V \subseteq U \cap U'$ such that $f \mid_U = f \mid'_U$. It is trivial to check that the so-defined relation $\sim$ is indeed an equivalence relation. Now consider the set of all pairs modulo this equivalence relation, i.e. $\mathcal{O}_X(U)/\sim$, by taking a typical element $f = \frac{g}{h}$ in $\mathcal{O}_X/\sim$ such that $f \neq 0$. Then $V = U \backslash Z(f) \cap U = U \backslash Z(g) \cap U \neq \emptyset$, which is clearly an open set. Now $(U - Z(g) \cap U, \frac{h}{g})$ has the property that on the open set $U - Z(f) \cap U$ $f \cdot \frac{h}{g} = 1$. Hence $(V, \frac{h}{g})$ serves as the inverse f^{-1} to (V, f). So we have a field, whose definition is given as follows:

Definition 1.4.4 *For a variety Y, its function field $K(Y)$ is defined as the collection of equivalence classes (U, f) where f is a regular function on an open set U, and two pairs (U, f), and (V, g) are considered equivalent in case $f = g$ on $U \cap V$.*

By $\mathcal{O}(Y)$ (or by $\mathcal{O}_Y$), and $\mathcal{O}_{Y,P}$ (or by $\mathcal{O}_P$) we denote the ring of all regular functions on a variety Y, and ring of germs of regular functions near P, for which we can give the formal definition as follows:

Definition 1.4.5 *Let $Y \subseteq \mathbb{A}^n$ be an affine variety. Then the ring defined by*

$$\mathcal{O}_{Y,P} = \{\frac{f}{g} \mid f, g \in k[x_1, \ldots, x_n] \quad and \quad g(P) \neq 0\} \subseteq K(Y)$$

*is called the **local ring** of Y at the point P. Evidently, the maps in $\mathcal{O}_{Y,P}$ can be considered as rational functions which are regular at P. If $U \subseteq Y$ is a non-empty subset, the ring of regular functions on U, denoted $\mathcal{O}_Y(U)$ is defined as*

$$\mathcal{O}_Y(U) = \bigcap_{P \in U} \mathcal{O}_{Y,P}$$

Remark: The ring $\mathcal{O}_{X,P}$ is a local ring, with maximal ideal $\mathfrak{m}_{X,P} = \{f = \frac{g}{h} \in \mathcal{O}_{X,P} \mid f(P) = 0\}$ of all functions that vanish at P. The ideal $\mathfrak{m}$ is maximal since any element $f = \frac{g}{h}$ not contained in $\mathfrak{m}$ has the property that $f(P) \neq 0$ hence $g(P) \neq 0$ and $h(P) \neq 0$. But then the function $\frac{h}{g}$ serves simply as the inverse of $f = \frac{g}{h}$. Hence $\mathcal{O}_{X,P} \backslash \mathfrak{m} \subseteq \mathcal{O}_{X,P}^{\times}$, where $\mathcal{O}_{X,P}^{\times}$ is the unit ring of $\mathcal{O}_{X,P}$. And conversely if $f = \frac{g}{h}$ is an element of $\mathcal{O}_{X,P}^{\times}$ then it cannot be equal to zero at P, so $\mathcal{O}_{X,P}^{\times} \subseteq \mathcal{O}_{X,P} \backslash \mathfrak{m}$. Hence the set of all non-units of the ring $\mathcal{O}_{X,P}$ is the ideal $\mathfrak{m}$, which proves that the ideal $\mathfrak{m}$ is maximal. It is easy to observe that $\mathcal{O}_{X,P}/\mathfrak{m} \cong k$, where the isomorphism is given by the evaluation of each element of $\mathcal{O}_{X,P}$ at the point P.

By naturally restricting the maps we have the injections $\mathcal{O}_P \hookrightarrow \mathcal{O}(Y) \hookrightarrow K(Y)$. By definiton of a morphism it is clear that for a variety Y, $\mathcal{O}(Y)$ and $K(Y)$ are invariants up to isomorphism.

A more subtle interpretation of "regular functions" on a variety can be given in terms of *sheafs*. To give this interpretation, we first briefly summarize what is a sheaf as follows:

Definition 1.4.6 *A presheaf $\mathcal{F}$ of rings on a topological space X consists of the data:*

- *for every open subset $U \subseteq X$ a ring $\mathcal{F}(U)$ (which can be considered as the ring of functions on U)*
- *for every inclusion $U \subseteq V$ of open sets in X, a ring homomorphism $\rho_{V,U}$: $\mathcal{F}(V) \to \mathcal{F}(U)$ called the restriction map (which can be considered as the usual restriction of functions to a subset)*

such that

- $\mathcal{F}(\emptyset) = 0$,
- $\rho_{U,U}$ *is the identity map on U,*
- *for any inclusion $U \subseteq V \subseteq W$ of open sets in X we have $\rho_{V,U} \circ \rho_{W,V} = \rho_{W,U}$*

The elements of $\mathcal{F}(U)$ are usually called the sections of $\mathcal{F}$ over U, and the restriction maps $\rho_{V,U}$ are written as $f \mapsto f\,|_U$.

A presheaf of rings is called a sheaf if it satisfies the additional glueing property: if $U \subseteq X$ is an open set, $\{U_i\}$ an open cover of U and $f_i \in \mathcal{F}(U_i)$ sections for all i such that $f_i\,|_{U_i \cap U_j} = f_j\,|_{U_i \cap U_j}$ for all i, j, then there is a *unique* $f \in \mathcal{F}(U)$ such that $f\,|_{U_i} = f_i$ for all i.

Example: If $X \subseteq \mathbb{A}^n$ is an affine variety, then the rings $\mathcal{O}_X(U)$ of regular functions on open subsets of X (with the obvious restriction maps $\mathcal{O}_X(V) \hookrightarrow \mathcal{O}_X(U)$ for $U \subseteq V$) form a sheaf of rings $\mathcal{O}_X$, the sheaf of regular functions or structure sheaf on X. In fact, defining properties of presheaves are obvious, and the glueing property of sheaves is easily seen from the description of regular functions.

1.5 Rational Maps, Birational Equivalence

Definition 1.5.1 *A rational map $\phi : X \to Y$ between two varieties X, and Y is an equivalence class of pairs (U, ϕ_U) with $\emptyset \neq U$ an open subset of X, and ϕ_U is a morphism of U to Y, where two pairs (U, ϕ_U) and (V, ϕ_V) are considered equivalent in case $\phi_U \mid_{U\cap V} = \phi_V \mid_{U\cap V}$. And a rational map ϕ is called* dominant *if $\phi_U(U)$ is dense in Y for some (U, ϕ_U).*

Observe that a set $A \subseteq Y$ is dense in Y if and only if $A \cap O \neq \emptyset$ for every open set $O \subseteq Y$, since otherwise $A \subseteq Y \backslash O \Rightarrow \overline{A} \subseteq Y \backslash O \subsetneqq Y \Rightarrow \overline{A} \neq Y$ and hence A cannot be dense. So, if a rational map $\phi : X \to Y$ is dominant then $\phi_U(U)$ is dense in Y for some and hence every $U \subseteq X$. To see this:

$$\phi_U(U) \quad \text{is dense in} \quad Y \Leftrightarrow \phi_U(U) \cap B \neq \emptyset \quad \text{for every open} \quad B \subseteq Y \Leftrightarrow$$

$$\phi_U^{-1}(B) \neq \emptyset \quad \text{in} \quad X \quad \text{for every open set} \quad B \subseteq Y \Leftrightarrow \text{ since } X \text{ is irreducible}$$

$$\phi_U^{-1}(B) \cap (U \cap V) \neq \emptyset \text{ for every open set } V \subseteq X \Leftrightarrow \phi_V^{-1}(B) \supseteq \phi_{U\cap V}^{-1}(B) \neq \emptyset$$

$$\Leftrightarrow \phi_V^{-1}(B) \neq \emptyset \quad \text{in X} \quad \Leftrightarrow \phi_V(V) \cap B \neq \emptyset \quad \text{for every open} \quad B \subseteq Y \Leftrightarrow$$

$$\phi_V(V) \quad \text{is dense in} \quad Y$$

Taking into account the contrapositive form of the statement, this definition of a *dominant rational map* is independent of which class ϕ_U is taken to check whether $\phi_U(U) \subseteq Y$ is dense, or not.

Definition 1.5.2 *A rational map with a rational inverse, i.e.* $\phi : X \to Y$ *for which* $\exists$ *a map* $\psi : Y \to X$ *such that* $\phi \circ \psi = id_Y$ *and* $\psi \circ \phi = id_X$ *as rational maps, is called a birational map. In case there is a birational map* $\phi : X \to Y$*, the varieties* X *and* Y *are called birationally equivalent, and sometimes birational in short.*

1.5.1 Isomorphism vs. Birational Equivalence

It is obvious that being isomorphic is a stronger concept than being birationally equivalent. In case there is an isomorphism $\phi : X \to Y$ between two varieties X and Y, then clearly this map ϕ with all open subsets U of X forms an equivalence class (ϕ, ϕ_U) where $\phi_U = \phi \mid_U$ is simply the restriction map. Then in accordance with the formal definition of a rational map, a morphism is naturally a rational map with the largest possible domain of definition, then an isomorphism appears trivially to be a birational equivalence.

On the other hand, two varieties X and Y are birationally equivalent if there are two open subsets $X' \subseteq X$ and $Y' \subseteq Y$ with the property that $X' \cong Y'$, i.e. X and Y possess isomorphic open subsets. In some cases this isomorphism cannot be extended to the whole variety, and these two varieties fail to be isomorphic. A very well-known counter-example for two birationally equivalent but non-isomorphic varieties is the following:

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$, $Y = \mathbb{P}^2$. Set theoretically $\mathbb{P}^1 \times \mathbb{P}^1 = \{([x_0 : x_1], [x_2, x_3]) : (x_0, x_1), (x_2, x_3) \in \mathbb{A}^2 \backslash \{(0,0)\} \quad \}$, and $\mathbb{P}^2 = \{[x_0 : x_1 : x_3] : (x_0, x_1, x_2) \in \mathbb{A}^3 \backslash \{(0,0,0)\} \quad \}$. Then let us define the following rational function

$$\phi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2 \textit{ such that}$$

$$\phi : ([x_0, x_1], [x_2 : x_3]) \mapsto [\frac{x_0}{x_3}, \frac{x_1}{x_3}, \frac{x_2}{x_3}], \quad \textit{when } x_3 \neq 0.$$

Since one of x_0, or x_1 must be nonzero and $x_3 \neq 0$, we must have $\frac{x_0}{x_3} \neq 0$ or

$\frac{x_1}{x_3} \neq 0$, thus the function is indeed well-defined. Also it is clear that the so-defined function ϕ has the maximal domain of definition $(\mathbb{P}^1 \times \mathbb{P}^1)\backslash H_{x_3}$, where $H_{x_3} = \{([x_0 : x_1], [x_2, x_3]) : x_3 = 0\}$ is the hyperplane generated by the monomial x_3. Thus, the maximal domain of definition for the function ϕ is an open subset of $\mathbb{P}^1 \times \mathbb{P}^1$, hence on any open subset U of $\mathbb{P}^1 \times \mathbb{P}^1$ the function is defined on the open set $U\backslash H_{x_3}$ of U and hence of $\mathbb{P}^1 \times \mathbb{P}^1$. Restricting to any open subset $V \subseteq (\mathbb{P}^1 \times \mathbb{P}^1)\backslash H_{x_3}$ we get our equivalence classes (V, ϕ_V), obtaining our rational map from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$. Moreover since $(x_0, x_1) \neq (0, 0)$, we conclude that the range of the map ϕ does not cover all of $\mathbb{P}^2$. Indeed it only misses the point $[0 : 0 : 1]$. Moreover $[0 : 0 : 1]$ is the zero locus of the irreducible polynomials x_0 and x_1, hence an algebraic set. Thus $\mathbb{P}^2\backslash\{[0 : 0 : 1]\}$ is an open subset of $\mathbb{P}^2$. It is easy to check that the inverse ϕ^{-1} of the function ϕ is defined as follows:

$$\phi^{-1} : \mathbb{P}^2\backslash\{[0 : 0 : 1]\} \to (\mathbb{P}^1 \times \mathbb{P}^1)\backslash H_{x_3}$$

$$\phi^{-1} : [x_0, x_1, x_2] \mapsto ([x_0, x_1], [x_2, 1])$$

Again it is trivial that

$$\mathbb{P}^1 \times \mathbb{P}^1 = H_{x_3} \bigcup ((\mathbb{P}^1 \times \mathbb{P}^1)\backslash H_{x_3})$$

where

$$H_{x_3} = \{([x_0 : x_1], [1 : 0]) : (x_0, x_1) \in \mathbb{A}^2\backslash\{(0, 0)\} \quad \}, \qquad \text{and}$$

$$(\mathbb{P}^1 \times \mathbb{P}^1)\backslash H_{x_3} = \{([x_0 : x_1], [x_2 : 1]) : (x_0, x_1) \in \mathbb{A}^2\backslash\{(0, 0)\} \quad \}$$

Since our function ϕ is defined on $\mathbb{P}^1 \times \mathbb{P}^1\backslash H_{x_3}$ only we can write it more simply as

$$\phi : (\mathbb{P}^1 \times \mathbb{P}^1)\backslash H_{x_3} \to \mathbb{P}^2\backslash\{[0 : 0 : 1]\}$$

$$\phi : ([x_0, x_1], [x_2, 1]) \mapsto [x_0, x_1, x_2]$$

Hence on their respective domains the functions ϕ and ϕ^{-1} are given as polynomials and therefore it is a trivial result that they must pull back the regular maps on their ranges to their domains of definition. Hence ϕ and ϕ^{-1} are bijective morphisms with respective domains $(\mathbb{P}^1 \times \mathbb{P}^1)\backslash H_{x_3}$ and $\mathbb{P}^2\backslash\{[0 : 0 : 1]\}$, therefore these two open subsets are isomorphic, i.e. $(\mathbb{P}^1 \times \mathbb{P}^1)\backslash H_{x_3} \cong \mathbb{P}^2\backslash\{[0 : 0 : 1]\}$. Moreover since the range of ϕ contains a nonempty open set, namely $\mathbb{P}^2\backslash\{[0:0:1]\}$, and any open set is dense in an irreducible Noetherian space; ϕ is a dominant rational map. But note that it cannot be extended to make it possible that $\mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{P}^2$. To see this, suppose by way of contradiction that there exists an isomorphism $\psi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$. It is an elementary fact that isomorphism of any two varieties is a bi-continuous map with respect to the Zariski topology on both sets, hence topologically a homeomorphism. Thus it must map open sets, and closed sets to open sets, and closed sets respectively. Consider the following closed subsets of $\mathbb{P}^1 \times \mathbb{P}^1$;

$$H_{x_0} = \{x_0 = 0\} = [0:1] \times \mathbb{P}^1, \quad and \quad H_{x_1} = \{x_1 = 0\} = [1:0] \times \mathbb{P}^1.$$

Clearly $H_{x_0} \cap H_{x_1} = \emptyset$. Since these two sets are closed in $\mathbb{P}^1 \times \mathbb{P}^1$, and ψ is a homeomorphism $\psi(H_{x_0}), \psi(H_{x_1}) \subseteq \mathbb{P}^2$ are closed sets in $\mathbb{P}^2$, in fact these are lines. But we know that any two lines (indeed any two curves) in $\mathbb{P}^2$ have nonempty intersection. Since dimension is a topological concept in Zariski topology, it is preserved by a homeomorphism. Then $\psi(H_{x_0})$ and $\psi(H_{x_1})$ must be a curve in $\mathbb{P}^2$, and hence they must have nonempty intersection, i.e. $\psi(H_{x_0}) \cap \psi(H_{x_1}) \neq \emptyset$ in $\mathbb{P}^2$. On the other hand, as ψ is bijective we must have $\psi(H_{x_0}) \cap \psi(H_{x_1}) = \psi(H_{x_0} \cap H_{x_1}) = \psi(\emptyset) = \emptyset$, contradiction to the previous result! Hence we conclude that there is no isomorphism between $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$, and thus $\mathbb{P}^1 \times \mathbb{P}^1 \ncong \mathbb{P}^2$. We will see in the following section that for curves the category of non-singular projective curves with dominant morphisms and the category of quasi-projective curves with dominant rational maps are equivalent.

1.6 Abstract Curves, Embedding in $\mathbb{P}^n$

1.6.1 Nonsingular Curves

The concept of a "regular value" is a very fruitful concept in differential geometry. In simplest terms, the value $q \in \mathbb{R}^m$ of a function $f = (f^1, \ldots, f^m) : \mathbb{R}^n \to \mathbb{R}^m$ is called a regular value in case the jacobian $\frac{\partial(f^1,\ldots,f^m)}{\partial(x_1,\ldots,x_n)} |_p$ has maximal rank at each $p = f^{-1}(q)$. This condition has a significance, because the pre-image $f^{-1}(q) \subseteq \mathbb{R}^n$ of a regular value q is always a $\mathcal{C}^\infty$ manifold with complementary dimension $rank(\frac{\partial(f^1,\ldots,f^m)}{\partial(x_1,\ldots,x_n)} |_{f^{-1}(q)})$ (e.g. $f : \mathbb{R}^{n+1} \to \mathbb{R}$ defined by $f(x_1, \ldots, x_{n+1}) = {x_1}^2 + \ldots + {x_{n+1}}^2$ in which case 1 is a regular value and $f^{-1}(1) = S^n$ is a $\mathcal{C}^\infty$ manifold of dimension n). Naturally the first definition of a nonsingular variety was given in terms of the partial derivatives of the generators of a variety, somehow using the jacobian concept in differential geomety.

Definition 1.6.1 *Let $X \subseteq \mathbb{A}^n$ be an affine variety with $f_1, \ldots, f_t \in k[x_1, \ldots, x_n]$ being a set of generators for the ideal $I(X)$. The variety X is said to be nonsingular at a point P if the matrix $\frac{\partial(f_1,\ldots,f_t)}{\partial(x_1,\ldots,x_n)} |_P$ has rank $n - r$ where $r = \dim X$. The variety X is said to be nonsingular in case it is nonsingular at each point.*

Later in a paper of Zariski [32] it has been shown that the concept of nonsingularity can be described intrinsically without looking at the way the variety is embedded in the affine space.

Definition 1.6.2 *Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and the corresponding residue class field $k = R/\mathfrak{m}$. The ring R is said to be a regular ring in case $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim R$. In general, $\dim_k \mathfrak{m}/\mathfrak{m}^2 \geq \dim R$.*

Depending on this definition, we cite the following theorem without proof. The theorem relates the way nonsingularity has been defined early to the concept of regularity for rings.

Theorem 1.6.1 *For an affine variety* $X \subseteq \mathbb{A}^n$ *with* $P \in X$ *the variety* X *is nonsingular at the point* $P \Leftrightarrow$ *the local ring* $\mathcal{O}_{X,P}$ *is a regular local ring.*

With this result of O. Zariski, the modern definition of nonsingularity for any kind of variety has transformed into the following format, expressing nonsingularity intrinsically :

Definition 1.6.3 *Let* X *be any variety with* $P \in X$*, then* X *is said to be nonsingular at the point* P *in case the local ring* $\mathcal{O}_{X,P}$ *is a regular local ring.* X *is said to be nonsingular in case it is nonsingular at each point it contains, and is said to be singular in case it is not nonsingular.*

1.6.2 Discrete Valuation Rings

Let K be a field. A discrete valuation on K is a function

$$v : K^{\times} \rightarrow \mathbb{R},$$

such that $v(K^{\times})$ is an abelian group of rank 1 and

$$v(xy) = v(x) + v(y), \qquad v(x+y) \geq min(v(x), v(y)).$$

Given v, define

$$R = R_v = \{r \in K : v(r) \geq 0\}, \qquad \mathfrak{m} = \mathfrak{m}_v = \{r \in K : v(r) > 0\}$$

Theorem 1.6.2 *: The ring* $(R,\mathfrak{m})$ *is a local ring (with maximal ideal* $\mathfrak{m}$*) of dimension 1. The ideal* $\mathfrak{m}$ *is principal, i.e.* $\mathfrak{m} = (\pi)$ *for some* $\pi \in R$*, and every other non-trivial ideal of* R *is of the form* (π^n) *for some* $n \geq 1$*.*

Proof: Let I be any ideal of R. Let r be any element of I such that $v(r)$ is minimal amongst the elements of I. We claim that $I = (r)$. One inclusion is clear. Let s be in I. Then $v(s/r) = v(s) - v(r) \geq 0$. Therefore $s/r \in R$ and hence $s = s \cdot s/r \in (r)$. Note that because every element a of $R/\mathfrak{m}$ has valuation zero the same holds for a^{-1}. Thus the units of R are precisely $R/\mathfrak{m}$ and therefore (R,$\mathfrak{m}$) is a local ring. Moreover, arguing as above, we see that if $I = (r)$ is an ideal and $v(r') \geq v(r)$ then $r' \in I$. That is, if $v(K^{\times}) = \alpha\mathbb{Z}$ for $\alpha > 0$, then the ideals of R are precisely the ideals

$$\{r \in R : v(r) \geq n\alpha\}$$

Taking $I = \mathfrak{m}$, we see that an element π such that $(\pi) = \mathfrak{m}$ exists. It is clear that $v(\pi) = \alpha$. Therefore, for any ideal I, a minimal element in I can be chosen as π^n. It follows also that R has a unique prime ideal. □

Definition 1.6.4 . *Let R be an integral domain with quotient field K. We say R is a discrete valuation ring (denoted DVR) if there exists a discrete valuation v on K such that $R = R_v$.*

Theorem 1.6.3 . *Let R be a local Noetherian integral domain of dimension 1. Then R is integrally closed if and only if R is a DVR.*

Proof: One direction is easy. If R is a DVR then R is integrally closed:

Let α be an element of the quotient field K that is integral over R. Write $\alpha = m/n$ where m and n are element of R. Then, for suitable $a_i \in R$ we have

$$(m/n)^s + a_{s-1}(m/n)^{s-1} + \ldots + a_0 \ = \ 0.$$

Without loss of generality assume $a_0 \neq 0$ (otherwise the monic polynomial annihilating (m/n) would be reducible). Now, in K we have the strong triangle inequality: $v(x+y) \ \geq \ min(v(x), v(y))$ with equality if $v(x) \neq v(y)$. If $v(m) < v(n)$

then one sees that $v(((m/n)^s + a_{s-1}(m/n)^{s-1} + \ldots + a_1) = s \cdot v(m/n) < 0$ while $v(-a_0) \geq 0$. Thus, $v(m) \geq v(n)$ and hence $\alpha = m/n$ is an element of R.

Conversely, assume that R is integrally closed local Noetherian domain of dimension 1. Let $\mathfrak{m}$ be unique maximal, hence prime ideal of R.

Step 1. $\mathfrak{m}$ is a principal ideal.

Let $a \in \mathfrak{m}$. For every $b \in R \backslash Ra$ we consider the ideal

$$(a : b) \;=\; \{r \in R : rb/a \;\in\; R\} \;=\; \{r \in R : rb \in Ra\}.$$

Choose b such that $(a : b)$ is maximal with respect to inclusion. We claim that $(a : b)$ is a prime ideal. Indeed, if $xy \in (a : b)$ and $x \notin (a : b)$ and $y \notin (a : b)$ (so $yb \notin Ra$). Then, since $x \in (a : yb)$ and $(a : yb) \supset (a : b)$ we get that $(a : b)$ is not maximal. Contradiction. Therefore, $(a : b)$ is prime. Now, since R is of dimension 1, $(a : b)$ is a maximal ideal, and since R is local $(a : b) = \mathfrak{m}$.

We next show that $\mathfrak{m} \;=\; R(a/b)$. First, $(b/a)\mathfrak{m} = R$, or, $\mathfrak{m} = R(a/b)$.

Step 2**.** Every ideal is a principal ideal.

Suppose not. Then we may take an ideal I which is maximal with respect to the property of not being principal (this uses noetherianity). We have $I \subset \mathfrak{m} = R\pi$. We get

$$I \subset \pi^{-1}I \subset R.$$

If $I = \pi^{-1}I$ then since I is a finitely generated R-module π^{-1} is integral over R, hence in R, hence $\mathfrak{m} = R$. Contradiction. It follows that $\pi^{-1}I$ strictly contains

I, therefore principal. But $\pi^{-1}I = (d)$ implies $I = (\pi d)$. Contradiction. Thus every ideal is principal.

Step 3. A principal local domain $(R, \mathfrak{m})$ is a DVR.
Let $\mathfrak{m} = (\pi)$. Define the function $v : R \to \mathbb{R}$ by, $v(x) = \max\{m \in \mathbb{Z} : x \in (\pi^m)\}$ for any $x \in R$. Note that in a principal ideal domain, the concepts of prime ($x \mid ab$ *implies* $x \mid a$ *or* $x \mid b$) and irreducible ($x = ab$ implies $a \in R^\times$ or $b \in R^\times$) are the same, and that a PID is a UFD. Pick an arbitrary element $x \in R$. Since R is a UFD, x can be factorized into prime (equivalently irreducible) elements. If π is a divisor of x, then $x = \pi^n \cdot x'$ such that $\pi \nmid x'$. Then clearly $(\pi^{n+1}) \subsetneqq (x) \subseteqq (\pi^n)$; thus $x \in (\pi^n)$, and $x \notin (\pi^{n+1})$; hence $v(x) = n$. If π is not a divisor of x, then $x \notin (\pi) = \mathfrak{m}$, the only maximal ideal of R; so we must have $(x) = R$, i.e. x is a unit. In this case, $\{m \in \mathbb{Z} : x \in (\pi^m)\} = \{0\}$, therefore $v(x) = 0$. So the above defined v is a well-defined function, and it is clear that $v(R) = \mathbb{Z}_{\geq 0}$. We only need to check that the two properties of a valuation are satisfied by v. Since R is a UFD, for arbitrary $x, y \in R$ there are unique non-negative integers m and n such that

$$x = \pi^n \cdot x_1 \qquad y = \pi^m \cdot y_1 \qquad \text{such that} \qquad \pi \nmid x_1 \text{ and } \pi \nmid y_1$$

So $xy = \pi^{n+m} \cdot x_1 \cdot y_1$, since π is a prime $\pi \nmid x_1 \cdot y_1$ by the above line. But then by our argumentation in the above paragraph, $v(xy) = n + m$. So $v(xy) = v(x) + v(y)$. Without loss of generality, assume in the above equality we have $n \geq m$ (since we a priori assume that our ring R is commutative), then $x + y = \pi^m \cdot (\pi^k \cdot x_1 + y_1)$ where $k = n - m \in \mathbb{Z}^+$. But this implies since $\pi^m \mid (x + y), \quad (x + y) \in (\pi^m)$ So by definition of v we have $v(x + y) \geq m = \min(v(x), v(y))$. Therefore the above defined v is indeed a discrete valuation defined on R. Definition of v easily (in the unique possible way) extends to the quotient field K of R with if $\frac{a}{b} \in K$ is an arbitrary element, then $v(\frac{a}{b}) = v(a) - v(b)$. By definition of v, $\forall x \in R \quad v(x) \geq 0$, hence $R \subseteq R_v$. For the opposite inclusion pick an arbitrary element $\frac{a}{b}$ of K with the property $\frac{a}{b} \in R_v$. Then as $a, b \in R$, and R is a UFD we can write $a = \pi^n \cdot a_1$ and $b = \pi^m \cdot b_1$ such that $\pi \nmid a_1$,

$\pi \nmid b_1$, and $n, m \in \mathbb{Z}_{\geq 0}$. Then $\frac{a}{b} = \pi^{n-m} \cdot \frac{a_1}{b_1}$. By $\frac{a}{b} \in R_v$, $v(\frac{a}{b}) = n - m \geq 0$, and $v(a_1) = v(b_1) = 0$ which means that $a_1, b_1 \notin (\pi) = \mathfrak{m}$. Since $\mathfrak{m}$ is the only maximal ideal of R, $R \backslash \mathfrak{m} = R^\times$, the units of R. Hence $a_1, b_1 \in R^\times$, so $\frac{a_1}{b_1} \in R^\times \subset R$. Then $\frac{a}{b} = \pi^{n-m} \cdot \frac{a_1}{b_1} \in R$. Hence $R_v \subseteq R$. Therefore $R_v = R$. In this case, v is a valuation with domain R and range $\mathbb{Z}_{\geq 0}$ such that $R_v = R$, which proves that R is a DVR. □

Corollary 1.6.1 *: Let R be a Dedekind ring. Then $R_\mathfrak{p}$ (the localization of R at $\mathfrak{p}$) is a DVR for every prime ideal $\mathfrak{p} \lhd R$.*

Proof: Clearly R is an integrally closed, local Noetherian ring with dimension 1, and $R_\mathfrak{p}$ is an integrally closed Noetherian ring with dimension 1, which is also a local ring with the only maximal ideal $\mathfrak{p}$. Then according to the above stated theorem, $R_\mathfrak{p}$ is a DVR. □

1.6.3 Curves

We have defined previously a curve over a field k as a variety (over k) of dimension 1.

If X is a curve, then for every regular point $p \in X$ (a point for which the maximal ideal of $\mathcal{O}_{X,p}$ is a regular local ring) the local ring $\mathcal{O}_{X,p}$ is a DVR. Note that since a DVR is a regular ring, if a point $p \in X$ has the property that $\mathcal{O}_{X,p}$ is a DVR (or integrally closed) then it is a regular point.

Now, if p is a regular point and, say, $X \subseteq \mathbb{A}^n$ (if needed, pass to an affine neighborhood), and $p = (p_1, \ldots, p_n)$, take a coordinate function $x_i - p_i$ on $\mathbb{A}^n$ that is not in $I(X)$. Then $x_i - p_i$ generates $\mathfrak{m}/\mathfrak{m}^2$. This shows that the discrete valuation of the local ring $\mathcal{O}_{X,p}$ is that of the order of vanishing of a function at the point p.

On the other hand, if p is a singular (non-regular) point then one cannot talk in general about the order of vanishing of a function at p in such terms. Indeed, if this is possible, we get that the local ring at p is a DVR and hence p is a regular point.

Theorem 1.6.4 (MAIN THEOREM) *The following categories are equivalent:*

(i) Non-singular projective curves and dominant morphisms.

(ii) Quasi-projective curves and dominant rational maps.

(iii) Function fields of transcendence degree 1 *over k and k-morphisms.*

We will prove this theorem on page 35 after some more algebraic preparation. Meanwhile note that the equivalence of (ii) and (iii) is already known to us. It is a special case of the equivalence between function fields and varieties up to birational equivalence. Also the transition from (i) to (ii) is quite clear. Every object of the first category is also an object of the second. Also every dominant morphism is a dominant rational map. Moreover, this functor of going from (i) to (ii) is faithful. That is, if two morphisms give the same birational map then they are equal to begin with. Indeed, the set where two morphisms are equal is closed, and if they agree as a rational map then it also contains a non-empty open set, thus equal to the whole curve.

Therefore, the new part in the theorem above is going from (ii) to (i). Namely, to associate to any quasi-projective curve $\mathcal{C}$ a non-singular projective curve $\widetilde{\mathcal{C}}$ in a canonical fashion, that depends only on the birational class of the initial curve, and to associate to every dominant rational map $f : \mathcal{C} \to \mathcal{D}$ a morphism $f : \widetilde{\mathcal{C}} \to \widetilde{\mathcal{D}}$, in a functorial way.

It is not hard to guess how $\widetilde{\mathcal{C}}$ should look like. If we take a projective closure $\mathcal{C}'$ of $\mathcal{C}$ in some projective space and let K be the function field of (the closure of) $\mathcal{C}$, then for every open affine set $U \subseteq \mathcal{C}'$ the preimage of U in $\widetilde{\mathcal{C}}$ should simply be the normalization of U. All those normalizations are done in the same field K and are compatible with intersections. Thus one hopes that there is a way to "glue" all of them together to a projective curve $\widetilde{C}$. The main point of what we are about to do is to show this is indeed possible. We remark that the gluing procedure itself, that is difficult from the point of view we are taking so far, becomes trivial in the category of schemes.

Let K/k be a *function field*. That is, K is a finitely generated field extension of k of transcendence degree one. Let C_K be the **set** of all discrete valuation rings of K/k. By that we mean a DVR, say R, contained in K, such that the valuation gives value zero to every nonzero element of k, and the quotient field of R is K, that is $Quot(R) = K$.

We shall attempt to view the set C_K itself as a curve. For that we need first to define a topology on C_K. We define a **topology** by taking the closed sets to be $\emptyset$, C_K, and every finite subset.

Before proceeding to define regular functions on open sets of C_K we immerse some algebraic results.

Lemma 1.6.1 `(MAIN LEMMA)` *For every $x \in K$ the set $\{R \in C_K : x \notin R\}$ is a finite set.*

Proof: Since the quotient field of R is equal to K for every $R \in C_K$, if $x \notin R$ then $x^{-1} \in \mathfrak{m}_R$. Thus, it is enough to prove that for every $y \neq 0$ the set

$$(y)_0 = \{R \in C_K : y \in \mathfrak{m}_R\}$$

is finite.

If $y \in k$ then $(y)_0$ is empty. Hence, we may assume that $y \notin k$. In this case, the ring $k[y]$ is a free polynomial ring and K is a finite extension of the field $k(y)$.

Let B be the integral closure of $k[y]$ in K. It is a finitely generated k-algebra (by Noether's theorem), integrally closed and of dimension 1. That is, B is a

Dedekind domain. Note that if $s \in K$ then s is algebraic over $k(y)$. Therefore, for some $g \in k[y]$ the element gs is integral over $k[y]$ (clear denominator in the minimal polynomial of s over $k(y)$). This shows that the quotient field of B is K. Therefore B defines a normal, hence non-singular, affine curve X with ring of regular functions B and function field K.

Now, suppose that $y \in R$ for some $R \in C_K$ then $k[y] \subseteq R$. Let $\mathfrak{m} = \mathfrak{m}_R$ be the maximal ideal of R and consider $\mathfrak{n} = \mathfrak{m} \cap B$. It is a prime, hence maximal, ideal of B. We have an inclusion of DVR's

$$B_{\mathfrak{n}} \subseteq B_{\mathfrak{m}}$$

with quotient field K. They must therefore be equal. Indeed if $A \subseteq B$ are two (nontrivial) DVR's with the same quotient field, say F, they must be equal. To see this, let v_A and v_B be the valuations on rings A and B respectively. If $A \neq B$ then by $A \subseteq B$ we must have $B \backslash A \neq \emptyset$. Hence $\exists\ 0 \neq b \in B \backslash A$. Then $v_B(b) \geq 0$, but $v_A(b) < 0$. Now for any $x \in F$ there exists $n \in \mathbb{N}$ such that $nv_A(b) < v_A(x)$, hence $v_A(b^n) < v_A(x) \Rightarrow v_A(xb^{-n}) > 0$, so $a = xb^{-n} \in A$ and therefore $x = ab^n \in A[b]$. Hence $F \subseteq \bigcup_{\{b \in B \backslash A\}} A[b] = B \subseteq F \Rightarrow B = F$. But then $\mathfrak{m}_B = \{non - units\ of\ B\} = \{0\}$, thus B is a trivial DVR, contradicting our assumption. Therefore we must have $A = B$ (Stating what we have proven differently: A subring V of a field is a nontrivial DVR $\Rightarrow V$ is a maximal subring of the field which is not a field itself).

We may more pleasantly rephrase what we proved as follows. Let $R \in C_K$ such that $y \in R$ then R is isomorphic to the local ring of some point x_R on X. (Thus every $R \in C_K$ is isomorphic to the local ring of some point on a non-singular affine curve with quotient field K!) If furthermore $y \in \mathfrak{m}_R$ then y, viewed as a function on X vanishes at x_R. That, for $y \neq 0$, can happen for only finitely many points. Hence, $\{R \in C_K : y \in \mathfrak{m}_R\}$ is a finite set. $\square$

Corollary 1.6.2

1. *Every $R \in C_K$ is isomorphic to the local ring of some point on a non-singular affine curve with quotient field K.*
2. *The set C_K is infinite, hence an irreducible topological space*
3. *For every $R \in C_K$ we have a canonical isomorphism $R/\mathfrak{m}_R = k$.*

Proof: The first claim was noted before. As for the second, the proof showed that all the local rings of X are elements of C_K. There are infinitely many such (if two points $x, y \in X$ define the same local ring, then the maximal ideals are equal. But the maximal ideals determine the point.) The last assertion follows immediately from the first. □

We may now define "`functions`" on C_K. Let $U \in C_K$ be a non-empty open set. We define

$$\mathcal{O}(U) \;=\; \bigcap_{R \in U} R.$$

We may make this more "function like" as follows. Every $f \in \mathcal{O}(U)$ defines a function

$$f : U \to k, \quad f(R) = f \quad (mod\ \mathfrak{m}_R).$$

If f and g are two elements of $\mathcal{O}(U)$ giving rise to the same function then $f - g \in \mathfrak{m}_R$ for any $R \in U$. Since C_K is infinite and U is not empty, U is infinite and therefore $f - g \in \mathfrak{m}_R$ for infinitely many $R \in C_K$. The main lemma implies that $f = g$.

Definition 1.6.5 *An abstract non-singular curve is an open subset U of C_K with induced topology and sheaf of regular functions.*

Let us now consider the category whose objects consist of all quasi-projective curves over k and all abstract non-singular curves. We define a morphism,

$$f : X \to Y,$$

between two objects of this category to be a continuous map of topological spaces, such that for every open subset $V \subseteq Y$, and every regular function $g : V \to k$, the composition

$$g \circ f : f^{-1}(V) \to k$$

is a regular function on $f^{-1}(V)$. There are no surprises in checking that this is a category. We may therefore speak on an isomorphism in this category.

More generally, given any object $\mathcal{C}$ in the above category, we define a morphism,

$$f : \mathcal{C} \to Y,$$

from $\mathcal{C}$ to a variety Y to be a continuous map, such that for every open set V in Y, and any regular function $g : V \to k$, the composition $g \circ f$ is a regular function of $f^{-1}(V)$.

Theorem 1.6.5 *Every non-singular quasi-projective curve Y is isomorphic to an abstract non-singular curve.*

Proof: It is pretty clear how to proceed. Let K/k be the function field of Y. Every local ring of a point $y \in Y$ is a DVR of K/k. Let $U \subseteq C_K$ be the set of the local rings of points of Y. Let $\phi : Y \to U$ be given by $\phi(y) = \mathcal{O}_{Y,y}$.

We first show that U is open. That is, that $C_K \backslash U$ is a finite set. If $Y' \subseteq Y$ is an open affine set, then it is enough to show that $C_K \backslash \phi(Y')$ contains finitely many points. We may therefore assume, to prove U is open, that Y is affine.

Let B be the affine coordinate ring of Y. It is a Dedekind ring with quotient field K and it is finitely generated over k. The proof of the main lemma shows that U consists precisely of all the DVR's of K/k that contain B. But if $x_1, \ldots, x_n$ are generators for B over k then $A \subseteq R$ for some $R \in C_k$ if and only if $x_1, \ldots, x_n$ belong to R. That is to say, if R is not in U then R does not contain at least one x_i and therefore

$$R \in \bigcup_{i=1,\ldots,n} \{R \in C_K : x_i \notin R\}.$$

The right hand side is a finite set by the main lemma.

By construction ϕ is a bijection. Moreover, a non-empty set in Y is open if and only if it is co-finite and the same holds in U, Thus (trivially) ϕ is bi-continuous. Moreover, if $V \subseteq Y$ is an open set then $\mathcal{O}(V) = \bigcap_{y\in V} \mathcal{O}_{Y,y} = \mathcal{O}(\phi(V))$. Thus, ϕ is an isomorphism. $\square$

Lemma 1.6.2 *Let X be an abstract non-singular curve, let $P \in X$, and let Y be a projective variety. Let*

$$\phi : X\backslash\{P\} \to Y$$

be a morphism. Then there exists a unique morphism,

$$\widetilde{\phi} : X \to Y,$$

extending ϕ.

Proof: The uniqueness of $\widetilde{\phi}$, if it exists, is clear: The set where two morphisms agree is closed.

To prove ϕ exists we may reduce to the case $Y = \mathbb{P}^n$. Indeed, since $Y \subseteq \mathbb{P}^n$ for some n, we may view ϕ as a morphism

$$\phi : X\backslash\{P\} \to \mathbb{P}^n.$$

If it extends to

$$\widetilde{\phi} : X \to \mathbb{P}^n,$$

then the preimage of Y under ϕ is a closed set containing $X\backslash\{P\}$, thus equal to X. That is, ϕ factors through Y.

Let therefore $\phi : X\backslash\{P\} \to P^n_{x_0,\ldots,x_n}$, be a morphism. Let

$$U = \{(x_0 : \cdots : x_n) : x_i \neq 0 \quad \forall i\}.$$

If $\phi(X\backslash\{P\}) \cap U = \emptyset$, then $\phi(X\backslash\{P\})$ being irreducible is contained in one of the hyperplanes $\{x_i = 0\}$ forming the complement of U. However, each such hyperplane is isomorphic to $\mathbb{P}^{n-1}$ and we are done by induction on the dimension. We may therefore assume that $\phi(X\backslash\{P\}) \cap U \neq \emptyset$. Therefore, for every i, j

the function $f_{ij} = \phi^*(x_i/x_j)$ is a regular function on $X\backslash\{P\}$. In particular, $f_{ij} \in K(X)$.

Let v_P be the valuation associated to the local ring P. Let

$$r_0 = v_P(f_{00}), r_1 = v_P(f_{10}), \ldots, r_n = v_P(f_{n0}).$$

Let i be an index such that r_i is minimal. Then, for every j we have

$$v_P(f_{ij}) = v_P(f_{j0}/f_{i0}) = r_j - r_i \geq 0.$$

Thus, $f_{ij} \in P$ for every j. We define

$$\widetilde{\phi}(P) = (f_{0i}(P), \ldots, f_{ni}(P)).$$

Note that this is well-defined! First $f_{ii} = 1$ and for every j we have $f_{ij}(P) \in k$.

To show that $\widetilde{\phi}$ is a morphism, it is enough to show that regular functions in a neighborhood of $\widetilde{\phi}(P)$ pull back to regular functions in a neighborhood of P. Note that in fact

$$\widetilde{\phi}(P) \in U_i = \{x : x_i \neq 0\} \cong \mathbb{A}^n_{\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}}.$$

It is enough to prove the assertion for open sets contained in U_i. Thus, it would be enough to show that $\widetilde{\phi}^*(x_j/x_i)$ is a regular function (the assertion then follows for any open set in U_i). But, at every point in the preimage of U_i that is not P this is already known and at P we have $\widetilde{\phi}^*(x_j/x_i) = f_{ji} \in P$. □

Theorem 1.6.6 *Let K/k be a function field. Then C_K is isomorphic to a non-singular projective curve.*

Proof: We saw that given $R \in C_K$ there exists some non-singular affine curves X_R and a point $x_R \in X_R$ such that $R \cong \mathcal{O}_{X,x_R}$. The curve X_R is isomorphic to the abstract curve $U \subseteq C_K$, where $U = \{\mathcal{O}_{X,x} : x \in X\}$. Therefore, we may write

$$C_K = \bigcup_R U_R,$$

where each U_R is isomorphic to an affine non-singular curve. However, since open sets are cofinite, C_K is quasi compact. Thus,

$$C_K = U_1 \cup \ldots \cup U_t,$$

where each U_i is an open affine subset, that is, isomorphic to a non-singular affine curve X_i. Say $\phi_i : U_i \to X_i$. Let Y_i be the closure of X_i in some projective space $\mathbb{P}^{n(i)}$. Applying the previous lemma successively, we see that there exists a morphism

$$\phi_i : C_K \to Y_i,$$

extending the one on U_i. Let

$$\phi : C_K \to Y_1 \times \ldots \times Y_t \subseteq \mathbb{P}^{n(1)} \times \ldots \times \mathbb{P}^{n(t)} \subseteq \mathbb{P}^N,$$

be the diagonal morphism. That is

$$\phi(R) = (\phi_1(R), \ldots, \phi_t(R)).$$

Let Y be the closure of the image of ϕ. It is a projective curve. We shall show that $\phi : C_K \to Y$ is an isomorphism.

Let $P \in C_K$. Then $P \in U_i$ for some i. Let $\pi : Y \to Y_i$ be the projection induced from $Y \subseteq \Pi Y_i$. Then $\pi \circ \phi = \phi_i$ on the set U_i. We get inclusions of local rings

$$\mathcal{O}_{Y_i,\phi_i(P)} \stackrel{\pi^*}{\to} \mathcal{O}_{Y,\phi(Y)} \stackrel{\phi^*}{\to} \mathcal{O}_{C_K,P}.$$

Moreover, since ϕ_i is an isomorphism on U_i, we get that all three rings are isomorphic ($\phi^* \circ \pi^*$ is an isomorphism). In particular, for every $P \in C_K$ the rings $\mathcal{O}_{Y,\phi(P)}$ and $\mathcal{O}_{C_K,P}$ are isomorphic under ϕ^*.

We next show ϕ is surjective. Let $Q \in Y$ and take some discrete valuation ring R containing $\mathcal{O}_{Y,Q}$ (localize the integral closure of $\mathcal{O}_{Y,Q}$ at a suitable prime ideal). Then R is the local ring of some point $P \in C_K$ and the argument above shows that $\mathcal{O}_{Y,\phi(P)}$ is isomorphic to R. If Q and Q' are points on a curve such that $\mathcal{O}_Q \subseteq \mathcal{O}_{Q'}$ then $Q = Q'$. Thus $\phi(P) = Q$ and therefore ϕ is surjective. This reasoning also shows that ϕ is injective, because $\mathcal{O}_{Y,\phi(P)} \cong \mathcal{O}_{C_K,P}$.

We got so far that ϕ is a bijective morphism such that ϕ^* induces an isomorphism of local rings. This implies that ϕ^{-1} is a morphism (use that the set where a function f on a variety Z is regular is precisely $\bigcup_{f \in \ \mathcal{O}_{Z,z}} z$). $\square$

Theorem 1.6.7 (MAIN THEOREM) *The following categories are equivalent:*

(i) Non-singular projective curves and dominant morphisms.

(ii) Quasi-projective curves and dominant rational maps.

(iii) Function fields of transcendence degree 1 *over k and k-morphisms.*

Proof: The functors (i)$\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are already known to us. We also know that (ii) $\Rightarrow$ (iii) is an equivalence of categories. It would therefore be enough to construct a functor (iii) $\Rightarrow$ (i) and show that (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) give an equivalence of categories.

Given a function field K/k associate to it the curve C_K. This curve is isomorphic to a non-singular projective curve. Given another function field K'/k and a homomorphism of k-algebras $K'/k \to K/k$ we have a rational map $C_K \to C_{K'}$, and therefore a morphism $U \to C_{K'}$ for some open non-empty set U in C_K. Thus, the morphism extends uniquely to a morphism $C_K \to C_{K'}$. It is immediate to verify that this process takes compositions to compositions, hence gives a functor (iii) $\Rightarrow$ (i).

Obviously, the objects associated to C_K and $C_{K'}$ under (i) $\Rightarrow$ (iii) are just K and K', and the induced map $K' \to K$ is just the one we have started with. Thus, the functors (i) $\to$ (iii) and (iii) $\to$ (i) are equivalence of categories. $\square$

1.7 Discrete Invariants: Degree and Genus

In order to show the motivation for the definition of genus, I will first give its definition as it is done in differential geometry.

By a regular region, we understand a region $R \subseteq S$ where R is compact with boundary ∂R being the finite union of (simple) closed piecewise regular curves which do not intersect, and S is a compact, connected, orientable 2-dimensional manifold. For convenience, we shall consider a compact 2-dimensional manifold as a regular region, whose boundary being empty. By a *triangle*, we mean a simple region which has only three vertices with external angles $\alpha_i \neq 0, \quad i = 1, 2, 3.$

A triangulation of a regular region $R \subseteq S$ is a finite family τ of triangles $T_i,\ i = 1, \ldots, n$, such that

1. $\bigcup_i^n T_i \ = \ R$
2. If $T_i \cap T_j \neq \phi$, then $T_i \cap T_j$ is either a common edge of T_i and T_j or a common vertex of T_i and T_j.

For a triangulation τ of a regular region $R \subseteq S$ of a surface S, we shall denote by F the number of triangles (faces), by E the number of sides (edges), and by V the number of vertices of the triangulation. The number

$$F \ - \ V \ + \ E \ = \ \chi$$

is called the *Euler-Poincare characteristic* of the triangulation. It is a celebrated theorem of differential geometry that every compact, connected, orientable 2-dimensional manifold admits a triangulation, and the *Euler-Poincare characteristic* of the manifold is independent of different choices of triangulation. Moreover again by the same theorem, all compact, connected, orientable 2-dimensional manifolds can be distinguished topologically by their *Euler-Poincare characteristic*, i.e. χ is unique up to homeomorphism, and each such manifold is homeomorphic to either S^2 (a sphere) or a sphere with a positive number of handles g. This number of handles g is called the `genus` of the manifold and related to the *Euler-Poincare characteristic* χ within the identity:

$$g \ = \ \frac{2 - \chi}{2}$$

Every complex algebraic curve is a compact, connected, orientable 2-dimensional topological manifold; and hence by the above-stated topological classification it is meaningful to talk about its *genus*. The following example illustrates the topological proof of *degree-genus formula* for plane curves.

Example: Let us now consider

$$C_d \;=\; \{(x,y)\in\mathbb{C}^2 \;\mid\; f(x,y)=0\}\subseteq\mathbb{C}^2\},$$

where f is an arbitrary polynomial of degree d. This is an equation that we certainly cannot solve easily to transform into form $y = g(x)$ in most cases. Perturbing the polynomial equation does not change the genus of the surface. Hence in order to examine the surface generated by the polynomial $f(x,y)$ it seems easier to deform the polynomial $f(x,y)$ to something singular which is easier to analyze. The easiest thing which shines in one's mind is to degenerate the polynomial f of degree d into a product of d linear equations $\mathcal{L}_1,\ldots,\mathcal{L}_d$:

$$C'_d \;=\; \{(x,y)\in\mathbb{C}^2 \;\mid\; \mathcal{L}_1\cdots\mathcal{L}_d=0\}\subseteq\mathbb{C}^2,$$

This surface should have the same "genus" as our original surface C_d.

It is not hard to see what C'_d looks like: undoubtedly it is just a union of n lines in $\mathbb{C}^2$. Any two of these lines intersect in a point, and we can certainly choose the lines so that no three of them intersect in a point. Every line after compactifying is just the complex sphere $\mathbb{C}_\infty$. And for example 3 lines chosen in this manner looks like 3 spheres with 3 total connections among where each sphere has a connection with the other remaining two. Now, we have d spheres, and every two of them connect in a pair of points, so in total we have $\binom{d}{2}$ connection. But $d-1$ of them are needed to glue the d spheres to a connected chain without loops; only the remaining ones then add a handle each. So the genus of C'_d (and hence of C_d) is

$$\binom{d}{2} - (d-1) \;=\; \binom{d-1}{2} \;=\; \frac{(d-1)(d-2)}{2}.$$

This formula is commonly known as the *degree-genus formula* for plane curves. We will derive the same formula more rigorously on the foregoing pages. □

Since nonsingular algebraic curves over $\mathbb{C}$ are all holomorphic manifolds, the above given topological definition of *genus* easily applies to them. But, for a nonsingular algebraic curve over an arbitrary field k there is no standard way to transform the underlying set of the curve to a topological manifold. In this case a more general notion of *genus* is given as follows.

Definition 1.7.1 *Let $\mathcal{C}$ be a nonsingular algebraic curve defined over a field k. Then the genus of $\mathcal{C}$ denoted $g(\mathcal{C})$, or g more briefly, is defined as*

$$g(\mathcal{C}) \;=\; \dim_k \, \Omega_{\mathcal{C}/k}$$

the dimension of the k-vector space of regular differential forms over $\mathcal{C}$, which is $\Omega_{\mathcal{C}/k}$.

To see that this definition of *genus* indeed coincides with the previous definition valid for nonsingular algebraic curves defined ove $\mathbb{C}$, first observe that a complex nonsingular algebraic curve is a 1 complex-dimenisonal, 2 real-dimensional $\mathcal{C}^\infty$ manifold. Moreover $H^0(\mathcal{C}, \Omega_{\mathcal{C}/\mathbb{C}}) \cong \mathbb{C}$. This definition of *genus* coincides with that of a complex algebraic curve. Such a complex algebraic curve $\mathcal{C}$ with $\dim_{\mathbb{C}} \Omega_{\mathcal{C}/\mathbb{C}} = g$ will have the Betti numbers

$$B_0 = 1$$

$$B_1 = 2g$$

$$B_2 = 1$$

and hence the "Euler-Poincare characteristic" $B_0 - B_1 + B_2 = 2 - 2g$. So both notions of "genus" coincide over $\mathbb{C}$. A detailed presentation of this coincidence is presented in [30], page 137.

Since any rational function defined on a nonsingular projective variety always extends globally, two nonsingular projective curves are birationally equivalent if and only if they are isomorphic. Furthermore letting $\Omega^m_{Y/k}(V)$ denote the differential m-forms defined over an open subset $V \subseteq Y$ of a curve Y, if there is a rational map $\phi : X \to Y$ then $\Omega^m_{X/k}(\phi^*(V)) \subseteq \Omega^m_{X/k}(\phi^{-1}(V))$. Hence in case there is a birational map $\phi : X \to Y$, $\Omega^m_{Y/k}(V) = \Omega^m_{X/k}(\phi^{-1}(V))$, and thus $\Omega^m_{Y/k} \cong \Omega^m_{X/k}$. That is to say, for nonsingular projective curves the genus g is a birational invariant.

1.7.1 Hilbert Polynomial

Hilbert polynomial of an algebraic variety is a very useful computational tool that can be used to read the *dimension*, *degree* and the *arithmetic genus* of the concerned variety. Before defining what it is we need to recall some algebraic definitions.

Definition 1.7.2 *A graded ring R is a ring with a direct sum decomposition*

$$R = \bigoplus_{i=0}^{\infty} R_i$$

where each of R_i is an abelian group such that $R_i R_j \subseteq R_{i+j}$ for all $i, j \geq 0$.

And the concept of a graded module is constructed upon the concept of a graded ring as depicted in the following definition.

Definition 1.7.3 *Let the ring $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded ring, an R-module M is called a graded R-module in case it has a direct sum decomposition:*

$$M = \bigoplus_{-\infty}^{\infty} M_i$$

where each of M_i is an abelian group and $R_i M_j \subseteq M_{i+j}$ for all i, j.

A trivial example of a graded ring is the polynomial ring $S = k[x_1, \ldots, x_n]$ with $S = S_0 \oplus S_1 \ldots$ where S_k is the set of homogeneous polynomials with degree k, in which case S is said to be graded by degree.

Definition 1.7.4 *Let M be a finitely generated graded R-module, where grading is carried out by degree. Then the function*

$$H_M(s) = dim_k(M_s)$$

is called the ***Hilbert function of*** M.

First of all, observe that this definition really makes sense. Since M is finitely generated over R by hypothesis, all of the dimensions $dim_k(M_s)$ must be finite. This follows from the fact that a finitely generated module over a Noetherian ring is itself a Noetherian module, and in turn every submodule of a Noetherian module is again Noetherian. It is a theorem due to Hilbert that for a finitely generated $k[x_0, \ldots, x_n]$-module M the Hilbert function $H_M(s)$ agrees for large values of s with a polynomial $P_M(s)$ of degree $\leq n-1$. Hence the information enclosed inside $H_M(s)$ can be read off by using finitely many values of $H_M(s)$ to construct the Hilbert polynomial $P_M(s)$.

The importance of Hilbert polynomial of homogeneous coordinate ring $S(X) = S/I(X)$ of a variety X is that degree d of P_M is exactly the dimension of the variety, and $d!$ times the leading coefficient of P_M is the degree of the variety.

Let $X \hookrightarrow \mathbb{P}^n$ be a projective variety embedded in $\mathbb{P}^n$, which has the Hilbert polynomial $P_X(n)$. Although the genus of a nonsingular projective curve $\mathcal{C}$ is defined as $g_{\mathcal{C}} = h^1(\mathcal{O}_{\mathcal{C}}) = \dim H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$, there are two other so-called notions of *genus* commonly appearing, and writing them down leads to the following list:

1. The genus g of a projective curve, defined by $g = h^1(\mathcal{O}_{\mathcal{C}}) = \dim H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$,
2. The arithmetic genus of a variety X denoted $p_a(X)$ defined by $p_a(X) = (-1)^{\dim X}(P_X(0) - 1)$,
3. The geometric genus p_g defined by $p_g = h^0(\omega_{\mathcal{C}}) = \dim H^0(\mathcal{C}, \omega_{\mathcal{C}})$.

For a nonsingular projective curve $\mathcal{C}$ all three notions of genus coincide. Hence as we talk about the genus of a nonsingular projective curve, we can mean any of the three genuses with no ambiguity.

1.8 Mapping Nonsingular Projective Curves into $\mathbb{P}^3$

In this section, we will show that any nonsingular projective curve embedded in $\mathbb{P}^N$ for some $N > 3$ can be mapped birationally (hence isomorphically) to $\mathbb{P}^3$ so that the image is still nonsingular with the same genus as the original curve in $\mathbb{P}^N$; and any further such projection to $\mathbb{P}^2$ results in a singular image with nodes as singularities.

Let $E \subseteq \mathbb{P}^n$ be a d-dimensional linear subspace with $n - d$ linearly independent defining equations $L_1 = \ldots = L_{n-d} = 0$, where each of L_i being a linear form. The *projection with center* E is defined as the rational map $\pi(x) = (L_1(x) : \ldots : L_{n-d}(x))$. It is trivial to check that this map is regular on $\mathbb{P}^n - E$, since at every point of $\mathbb{P}^n - E$ at least one of the L_i does not vanish. Thus for any closed subvariety $X \subseteq \mathbb{P}^n$ which satisfies the condition $X \cap E = \emptyset$, the restriction $\pi|_X$ defines a regular map $\pi|_X : X \to \mathbb{P}^{n-d-1}$. Geometrically, if we identify any $n-d-1$-dimensional linear subspace $H \subseteq \mathbb{P}^n$ subject to the condition $E \cap H = \emptyset$ with $\mathbb{P}^{n-d-1}$, this means that there is a unique $d+1$-dimensional linear subspace passing through E and any point $x \in \mathbb{P}^n \backslash E$, whose intersection with H resulting in a unique point that is $\pi(x)$.

Lemma 1.8.1 *If $X \subseteq \mathbb{P}^n$ is a closed subvariety, and $E \subseteq \mathbb{P}^n$ is a linear subspace satisfying the condition $E \cap H = \emptyset$; then the projection $\pi : X \to \mathbb{P}^{n-d-1}$ with center E defines a finite map $X \to \pi(X)$.*

Proof: Let us denote the homogeneous coordinates on $\mathbb{P}^{n-d-1}$ by $y_0, \ldots, y_{n-d-1}$. As in conjunction with our preceding definition, suppose that π is given by $y_j = L_j(x)$ for $j = 0, \ldots, n-d-1$, where $x \in X$. Clearly $U_i = \pi^{-1}(\mathbb{A}_i^{n-d-1} \cap X)$ is given by the condition $L_i(x) \neq 0$, and is an affine open subset of X. We will show that $\pi : U_i \to \mathbb{A}_i^{n-d-1} \cap \pi(X)$ is a finite map. Any function $g \in k[U_i]$ is of the form $g = \frac{G_i(x_0,\ldots,x_n)}{L_i^m}$, where G_i is a form of degree m. Consider the map $\pi_1 : X \to \mathbb{P}^{n-d}$ defined by $z_j = L_j^m(x)$ for $j = 0, \ldots, n-d-1$ and $z_{n-d} = G_i(x)$, where $z_0, \ldots, z_{n-d}$ are homogeneous coordinates in $\mathbb{P}^{n-d}$. This is trivially a regular map, and hence its image $\pi_1(X) \subseteq \mathbb{P}^{n-d}$ is closed. Suppose that $\pi_1(X)$ is given by equations $F_1 = \ldots = F_s = 0$.

Since $X \cap E = \emptyset$, the forms L_i for $i = 0, \ldots, n-d-1$ share no common zeros on X. Thus the point $O = (0 : \ldots : 0 : 1) \in \mathbb{P}^{n-d}$ is not contained in $\pi_1(X)$, or saying differently the equations $z_0 = \ldots = z_{n-d-1} = F_1 = \ldots = F_s = 0$ do not have solutions in $\mathbb{P}^{n-d}$. Now consider the homogeneous ideal $\mathfrak{U}$ whose generators are $z_0, \ldots, z_{n-d-1}, F_1, \ldots, F_s$. Defining $t_i = z_j/z_0$ and passing to affine coordinates to express the generators $z_0, \ldots, z_{n-d-1}, F_1, \ldots, F_s$ as polynomials in $t_0, \ldots, t_{n-d}$, we see that the generators do not have a common root. Indeed, a common root $(\alpha_1, \ldots, \alpha_{n-d})$ would give a common root $(1, \alpha_1, \ldots, \alpha_{n-d})$ for the generators of $\mathfrak{U}$. By Hilbert's Nullstellensatz there must exist polynomials $H_j(t_0, \ldots, t_{n-d})$ such that $\sum_i z_i(1, t_1, \ldots, t_{n-d}) H_i(t_1, \ldots, t_{n-d-1}) + \sum_k F_k(1, t_1, \ldots, t_{n-d}) H_k(t_1, \ldots, t_{n-d-1}) = 1$. Now writing $t_j = \frac{z_j}{z_0}$ in this equality and multiplying through by a common denominator of the form $z_0^{l_0}$ we obtain that $z_0 \in \mathfrak{U}$. In the same way, for each $i = 1, \ldots, n$ there exists a number $l_i \in \mathbb{Z}^+$ such that $z_i^{l_i} \in \mathfrak{U}$. Letting $l = max(l_0, \ldots, l_{n-d})$ and $k = (l-1)(n-d+1)+1$ then in any term $z_0^{a_0} \cdots z_{n-d}^{a_{n-d}}$ of degree $a_0 + \ldots + a_{n-d} \geq k$ we must have by pigeonhole principle at least one term z_i with exponent $a_i \geq k \geq l_i$, and since $z_i^{l_i} \in \mathfrak{U}$, this term is contained in $\mathfrak{U}$. Hence denoting the ideal of $k[z_0, \ldots, z_{n-d}]$ which contains polynomials having only terms of degree $\geq k$ by I_k, we conclude that $(z_0, \ldots, z_{n-d-1}, F_1, \ldots, F_s) \supseteq I_k$. In particular,

$z_{n-d}^k \in (z_0, \ldots, z_{n-d-1}, F_1, \ldots, F_s)$. This means that we can write

$$z_{n-d}^k = \sum_{j=0}^{n-d-1} z_j H_j + \sum_{j=1}^{s} F_j P_j$$

where P_j and H_j are some polynomials. Writing $H^{(q)}$ for the homogeneous component of H of degree q, we observe that

$$\phi(z_0, \ldots, z_{n-d}) = z_{n-d}^k - \sum z_j H_j^{(k-1)} = 0 \qquad \text{on} \quad \pi_1(X).$$

The homogeneous polynomial ϕ has degree k and as a polynomial in the variable z_{n-d} it also has leading coefficient 1:

$$\phi = z_{n-d}^k - \sum_{j=0}^{k-1} A_{k-j}(z_0, \ldots, z_{n-d-1}) z_{n-d}^j$$

Substituting the formulas defining the map π_1 in the equation of ϕ regarding its annihilation on $\pi_1(X)$ we obtain $\phi(L_0^m, \ldots, L_{n-d-1}^m, G_i) = 0$ on X, with ϕ of the just above form. Dividing this relation by $L_i^{m_k}$ we get the required relation

$$g^k - \sum_{j=0}^{k-1} A_{k-j}(x_0^m, \ldots, 1, \ldots, x_{n-d-1}^m) g^j = 0,$$

where $x_r = \frac{y_r}{y_i}$ are coordinates in $\mathbb{A}_0^{n-d-1}$. Hence the lemma is proved. $\square$

Before ongoing to the next lemma, we will need to recall a result in commutative algebra called the "*Nakayama's Lemma*‘" which is merely stated as follows.

Lemma 1.8.2 *(Nakayama's Lemma) Let M be a finitely generated module over a ring R and $\mathfrak{I} \subseteq R$ an ideal. Suppose that for any element $a \in 1 + \mathfrak{I}$, $aM = 0$ implies that $M = 0$. Then $\mathfrak{I}M = M$ implies the result $M = 0$. Moreover; if $m_1, \ldots, m_n \in M$ have images in $M/\mathfrak{I}M$ that generate it as an $R-$module, then $m_1, \ldots, m_n$ generate M as an $R-$module.*

Lemma 1.8.3 *A finite map f is an isomorphic embedding $\Longleftrightarrow$ f is injective and its differential $d_p f$ is an isomorphic embedding of the Zariski tangent space $T_p(X) = (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$ for every $p \in X$.*

Proof: One direction is trivial; an isomorphism is always injective and since $f \circ f^{-1} = \mathrm{id}|_X$, we have that $d(f \circ f^{-1}) = df \circ df^{-1} = d(\mathrm{id}|_X) = \mathrm{id}|_{T_p(X)}$. Moreover since the Zariski tangent space $T_p(X)$ is a vector space, we conclude that $d_p f$ is an isomorphic embedding of the Zariski tangent space $T_p(X)$ for every $p \in X$. For the other direction, let us begin by setting $f(X) = Y$ and $\varphi = f^{-1}$. We are done if we show that φ^{-1} is a regular map, which is a local assertion. For $y \in Y$, let $x \in X$ be such that $f(x) = y$. Writing U and V for affine neighborhoods of x and y with $f(U) = V$ and such that $k[U]$ is integral over $k[V]$, let us also write f for the restriction $f|_U$. It is sufficient to prove that f is an isomorphism for appropriate choice of U and V, since then $\varphi = f^{-1}$ is a regular map at y.

Our second hypothesis is that $f^* : \mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$ is surjective. In other words, if $\mathfrak{m}_y = (u_1, \ldots, u_k)$, then the elements $f^*(u_i) + \mathfrak{m}_x^2$ generate $\mathfrak{m}_x/\mathfrak{m}_x^2$. Applying Nakayama's lemma to $\mathfrak{m}_x$ as an $\mathcal{O}_X$-module yields that $\mathfrak{m}_x = (f^*(u_1), \ldots, f^*(u_k))$, or in other words

$$\mathfrak{m}_x = f^*(\mathfrak{m}_y)\mathcal{O}_x.$$

We verify that $\mathcal{O}_x$ is a finite module over $f^*(\mathcal{O}_y)$. Since $k[U]$ is a finite $k[V]$-module, it is enough to prove that each element of the module $\mathcal{O}_x$ can be expressed in the form $\frac{\xi}{f^*(a)}$ with $\xi \in k[U]$ and $a \notin \mathfrak{m}_y$. To show this, it is enough to check that for any $\alpha \in k[U]$ with $\alpha \notin \mathfrak{m}_x$ there exists an element $a \in k[V]$ with $a \notin \mathfrak{m}_y$ such that $f^*(a) = \alpha\beta$ with $\beta \in k[U]$. Since finite maps take closed sets to closed sets, $f(V(\alpha))$ is closed, and since f is one-to-one $y \notin f(V(\alpha))$. Hence there exists a function $c \in k[V]$ such that $c = 0$ on $f(V(\alpha))$ and $c(y) \neq 0$. Then $f^*(c) = 0$ on $V(\alpha)$ and $f^*(c)(x) \neq 0$. By Hilbert's Nullstellensatz $f^*(c^n) = \alpha\beta$ for some $n > 0$ and $\beta \in k[U]$. We can set $a = c^n$.

The equality $\mathfrak{m}_x = f^*(\mathfrak{m}_y)\mathcal{O}_x$ shows that $\mathcal{O}_x/f^*(\mathfrak{m}_y)\mathcal{O}_x = \mathcal{O}_x/\mathfrak{m}_x = k$, and hence is generated by the single element 1. Applying Nakayama's lemma to $\mathcal{O}_x$ as a $f^*(\mathcal{O}_y)$-module yields that $\mathcal{O}_x = f^*(\mathcal{O}_y)$.

Let $u_1, \ldots, u_l$ be a basis of $k[U]$ as a module over $k[V]$. By what has been proven, $u_i \in \mathcal{O}_x = f^*(\mathcal{O}_y)$. Writing $V' = V \backslash V(h)$ for a principal affine neighborhood of y such that all $(f^*)^{-1}(u_i)$ are regular in $U' = U \backslash V(f^*(h))$. Then $k[U'] = \sum f^* k[V'] u_i$. By assumption $u_i \in f^*(k[V'])$, and it follows that

$k[U'] = k[V']$, which means that $f : U' \to V'$ is an isomorphism, thereby proving the assertion of the lemma. □

Theorem 1.8.1 *Let $X \subseteq \mathbb{P}^N$ be a variety and $\xi \in \mathbb{P}^N \backslash X$. Suppose that every line through ξ intersects X in at most one point, and ξ is not contained in the tangent space to X at any point, then the projection from ξ is an isomorphic embedding $X \hookrightarrow \mathbb{P}^{N-1}$.*

Proof: By previous lemmas the projection with center ξ defines a finite map, moreover by the assumptions set on the intersection of X and lines through ξ the projection map is injective. Since the projection from ξ is a map whose coordinate components are all linear functions, the differential of this projection map will trivially be an isomorphic embedding. By the previous lemma therefore the so-called projection from ξ is an isomorphic embedding $X \hookrightarrow \mathbb{P}^{N-1}$.

□

Theorem 1.8.2 *A nonsingular projective n-dimensional variety is isomorphic to a subvariety of $\mathbb{P}^{2n+1}$.*

Proof: It is sufficient to show that if $X \subseteq \mathbb{P}^N$ is a nonsingular n-dimensional variety and $N > 2n+1$ then there exists a specific point ξ which satisfies the hypothesis of the previous corollary; this is a standard dimension count. Let U_1 and $U_2 \subseteq \mathbb{P}^N$ be the sets consisting of points $\xi \in \mathbb{P}^N$ such that ξ does not satisfy the two assumptions of our previous corollary.

In $\mathbb{P}^N \times X \times X$ consider the set Γ of triples (a, b, c) with $a \in \mathbb{P}^N$, $b, c \in X$ such that a, b, c are collinear. Γ is trivially seen to be a closed subset of $\mathbb{P}^N \times X \times X$. The projections of $\mathbb{P}^N \times X \times X$ to $\mathbb{P}^N$ and to $X \times X$ define regular maps $\varphi : \Gamma \to \mathbb{P}^N$ and $\psi : \Gamma \to X \times X$. Obviously if $y \in X \times X$ with $y = (b, c)$, and $b \neq c$ then $\psi^{-1}(y)$ consists of points $(a, b, c,)$ where a is any point of the line through b and c. Hence dim $\psi^{-1}(y) = 1$, and so dim $\Gamma = 2n+1$. By definition $U_1 = \varphi(\Gamma)$, and hence dim $U_1 \leq$ dim $\Gamma = 2n+1$.

In exactly the same fashion, to study the set U_2 consider $\mathbb{P}^N \times X$ and the set Γ' consisting of points (a, b) such that $a \in T_b(X)$. Again we have projections $\psi : \Gamma' \to X$ and $\varphi : \Gamma' \to \mathbb{P}^N$. For $b \in X$ we have dim $\psi^{-1}(b) = n$ since X is nonsingular, and hence dim $\Gamma' = 2n$, and since $U_2 = \varphi(\Gamma')$, also dim $U_2 \leq 2n$.

To sum up, dim $U_1 \leq$ dim $\Gamma = 2n+1$ and dim $U_2 \leq 2n$; therefore if $N > 2n+1$ then $U_1 \cap U_2 \neq \mathbb{P}^N$, which is the required result to be proven. □

Corollary 1.8.1 *Any nonsingular quasiprojective curve is isomorphic to a curve in $\mathbb{P}^3$.*

Proof: Since all curves have dimension 1, the result follows by the preceding theorem for $n = 1$. □

As we have stated that every nonsingular abstract curve can be embedded in some $\mathbb{P}^N$, and every nonsingular algebraic curve in any $\mathbb{P}^N$ for some $N \geq 4$ can be projected into $\mathbb{P}^3$ isomorphically; we must also state that this process cannot be kept on further to obtain an embedding in $\mathbb{P}^2$. For this purpose we simply state the following theorem whose detailed proof can be found in [13], pp. 314-315.

Theorem 1.8.3 *Let $\mathcal{C}$ be a curve in $\mathbb{P}^3$. Then there is a point $O \notin \mathcal{C}$ such that the projection from the point O determines a birational morphism φ from $\mathcal{C}$ to its image in $\mathbb{P}^2$, and that image has at most nodes for singularities.*

Now by this theorem, we know that any curve is birationally equivalent to a plane curve possessing nodes at most as singularities. We also note that, by the following remarks, if a curve lying in $\mathbb{P}^3$ can be projected into $\mathbb{P}^2$ birationally then the resulting image cannot be nonsingular. Hence $\mathbb{P}^3$ is the best possible projective space into which a nonsingular algebraic curve can be embedded.

Remark: Although we have proven that any nonsingular projective curve lying in any $\mathbb{P}^n$ with $n \geq 3$ can be mapped birationally into $\mathbb{P}^3$, preserving nonsingularity and the genus, we can also show that continuing this projection, the process no longer extends to a mapping into $\mathbb{P}^2$ with a nonsingular image. Indeed, letting $X \subseteq \mathbb{P}^3$ be a curve in $\mathbb{P}^3$ which is not contained in any plane (and hence not certainly a curve in $\mathbb{P}^2$). Now if $O \notin X$ is a point, such that the projection from O induces a birational morphism φ from X to its image in $\mathbb{P}^2$, $\varphi(X)$ must be singular. To see this result we argumentate as follows. First, notice that since X is contained in no plane and φ is a projection from a point, $\varphi(X)$ is contained in no line in $\mathbb{P}^2$. By way of contradiction, assume that $Y = \varphi(X)$ is not singular. Then X is isomorphic to Y since they are birational. Associate X with its image to simplify notation. Use the twisted exact sequence (for $n = 2$ and $n = 3$) $0 \to \mathcal{I}_X(1) \to \mathcal{O}_{\mathbb{P}^n}(1) \to \mathcal{O}_X(1) \to 0$, where $\mathcal{I}_X$ is the ideal sheaf defining $X \subseteq \mathbb{P}^3$, and compute the long exact sequence of cohomology. For $n = 2$ and $n = 3$ we have that $H^0(\mathbb{P}^3, \mathcal{I}_X(1)) = 0$. Indeed, if there are any global sections of degree 1, there is a linear polynomial in the both ideals defining X, and then X is contained in a plane ($n = 3$) or a line ($n = 2$), both being contradictions. Thus we have $0 \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^n}(1)) \to H^0(X, \mathcal{O}_X(1)) \to H^1(\mathbb{P}^3, \mathcal{I}_X(1)) \to \dots$ (Note that the last map of H^0's is not surjective in general. As a simple example, take something not projectively normal, like the quadratic embedding $(s : t) \hookrightarrow (s^4 : s^3t : st^3 : t^4)$.) since the first term has dimension $n + 1$, we get for $n = 3$ that $\dim H^0(\mathbb{P}^3, \mathcal{O}_X(1)) \geq 4$. In case $n = 2$ if we show that $\dim H^1(X, \mathcal{I}_X(1)) = 0$, then we get $\dim H^0(\mathbb{P}^2, \mathcal{O}_X(1)) = 3$, a contradiction (Note that this argument assumes that $\mathcal{O}_X(1)$'s are the same over $\mathbb{P}^2$ and $\mathbb{P}^3$, since the sheaf is the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$).

For $n = 2$, the curve is a Cartier divisor, so $\mathcal{I}_X \cong \mathcal{L}(-D)$. Now $\mathcal{L} \in \text{Pic } \mathbb{P}^n \Rightarrow \mathcal{L} \cong \mathcal{O}_X(n)$, for some $n \in \mathbb{Z}$. Then $D \sim dH$, (H any hyperplane in $\mathbb{P}^n$) so we combine to get $\mathcal{I}_X \cong \mathcal{L}(-D) \cong \mathcal{L}(-dH) \cong \mathcal{O}_{\mathbb{P}^2}(-d)$. Twisting, $\mathcal{I}_X(1) \cong \mathcal{O}_{\mathbb{P}^2}(1 - d)$. $d = 1 \Rightarrow Y \cong \mathbb{P}^1$, but Y is not contained in any line; so contradiction! Thus $d > 1$, and $H^1(\mathbb{P}^2, \mathcal{I}_X(1)) \cong H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) = 0$, which is the desired result to be proven which leads to the contradiction $\dim H^0(\mathbb{P}^2, \mathcal{O}_X(1)) = 3$. Therefore $\varphi(X) \subseteq \mathbb{P}^2$ cannot be nonsingular, hence it must be a singular curve.

Remark: Although the projection of a nonsingular curve in $\mathbb{P}^3$ into $\mathbb{P}^2$ results in a plane curve with nodes in $\mathbb{P}^2$, the converse is not true that every plane curve with nodes is a projection of a nonsingular curve in $\mathbb{P}^3$. An easy counterexample is the curve with equation $xy+x^4+y^4=0$ (char $k \neq 0$). This curve has the only singularity at $(0,0)$ which is a node. Suppose by way of contradiction that this curve X is the projection of a nonsingular curve $\widetilde{X} \subseteq \mathbb{P}^3$, so there is a morphism $f : \widetilde{X} \to X$. Then there is an exact sequence

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{\widetilde{X}} \to \sum_{P\in X} \widetilde{\mathcal{O}_P}/\mathcal{O}_P \to 0,$$

where $\widetilde{\mathcal{O}_P}$ is the integral closure of $\mathcal{O}_P$. For each $P \in X$, let $\delta_P = \text{length}(\widetilde{\mathcal{O}_P}/\mathcal{O}_P)$. Now we claim that $p_a(X) = p_a(\widetilde{X}) + \sum_{P\in X} \delta_P$. First, note that since the projection of $\widetilde{X}$ into X is an affine morphism of Noetherian separated schemes; for all $i \geq 0$ there are natural isomorphisms $H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = H^i(X, f_*\mathcal{O}_{\widetilde{X}})$. Moreover since X has dimension 1 by Grothendieck's vanishing theorem for all $i \geq 2$ $H^i(X,\cdot) = 0$. Second, taking global sections and applying the exact sequence of cohomology gives an exact sequence

$$0 \to H^0(X,\mathcal{O}_X) \to H^0(X, f_*\mathcal{O}_{\widetilde{X}}) \to H^0(X, \sum_{P\in X} \widetilde{\mathcal{O}_P}/\mathcal{O}_P)$$
$$\to H^1(X,\mathcal{O}_X) \to H^1(X, f_*\mathcal{O}_{\widetilde{X}}) \to H^1(X, \sum_{P\in X} \widetilde{\mathcal{O}_P}/\mathcal{O}_P) = 0.$$

And by $H^i(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = H^i(X, f_*\mathcal{O}_{\widetilde{X}})$, and $H^0(X, \sum_{P\in X} \widetilde{\mathcal{O}_P}/\mathcal{O}_P) = \sum_{P\in X} \widetilde{\mathcal{O}_P}/\mathcal{O}_P$ this exact sequence becomes

$$0 \to H^0(X,\mathcal{O}_X) \to H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \to \sum_{P\in X} \widetilde{\mathcal{O}_P}/\mathcal{O}_P$$
$$\to H^1(X,\mathcal{O}_X) \to H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) \to 0.$$

Hence computing the dimensions of the $k-$modules appearing in the above exact sequence we have

$$\dim_k H^0(X,\mathcal{O}_X) - \dim_k H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) + \dim_k \sum_{P\in X} \widetilde{\mathcal{O}_P}/\mathcal{O}_P$$
$$-\dim_k H^1(X,\mathcal{O}_X) + \dim_k H^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = 0 \Leftrightarrow$$

$$1-1+\sum_{P\in X}\dim_k \widetilde{\mathcal{O}_P}/\mathcal{O}_P - p_a(X)+p_a(\widetilde{X}) = 0 \Leftrightarrow \sum_{P\in X}\delta_P - p_a(X)+p_a(\widetilde{X}) = 0 \Leftrightarrow$$

$$p_a(X) = p_a(\widetilde{X}) + \sum_{P\in X}\delta_P.$$

The invariant δ_P depends only on the analytic isomorphism equivalence class of the singularity at P. Hence computing δ_P for the node and cusp of suitable plane cubic curves, we see that $\delta_P = 1$ if the point P is a node or an ordinary cusp.

Now by above consideration $g(X) = \frac{1}{2}(d-1)(d-2)-r = \frac{1}{2}(3)(2)-1 = 2$, but no such curve exists. To see the last assertion, we simply give a nice classification of degree-4 curves sitting in any $\mathbb{P}^n$. If X is a curve of degree 4 in some $\mathbb{P}^n$ either

1. $g = 0$, in which case X is either the rational normal quartic in $\mathbb{P}^4$ or the rational quartic curve in $\mathbb{P}^3$, or
2. $X \subseteq \mathbb{P}^2$, in which case $g = 3$, or
3. $X \subseteq \mathbb{P}^3$, and $g = 1$.

To prove our observation;

(1.) if $g(X) = 0$ then X is isomorphic to $\mathbb{P}^1$. $X \nsubseteq \mathbb{P}^2$, since in $\mathbb{P}^2$ we have the genus implied by Plücker's genus formula $g = \frac{1}{2}(d-1)(d-2) = 3 \neq 0$. Then we claim that X lies either in $\mathbb{P}^3$ or $\mathbb{P}^4$. Indeed, if X is any curve of degree d in some $\mathbb{P}^n$, with $d \leq n$, which is not contained in any $\mathbb{P}^{n-1}$, we must have $d = n$, $g(X) = 0$, and X differs from the rational normal curve of degree d only by an automorphism of $\mathbb{P}^d$. To see this result, take a hyperplane H in $\mathbb{P}^n$. Then $H \cdot X$ consists of d points (counted with multiplicity). These points span a hyperplane of dimension $d-1$. If $d < n$, then we can add any other points on X until a hyperplane of dimension $n-1$ is spanned, but this new hyperplane contains H, so H must itself have intersected X in $n-1$ points, contradicting $d < n$. Thus we must have $d = n$. Thus, since X had degree 4, X is either contained in $\mathbb{P}^4$ for sure by the above argument. X cannot lie in $\mathbb{P}^1$ or $\mathbb{P}^2$, so we are left with only two possibilities either $X \subseteq \mathbb{P}^3$ or $X \subseteq \mathbb{P}^4$. Let us first consider the case $X \subseteq \mathbb{P}^3$.

By a *nonsingular rational quartic curve* in $\mathbb{P}^3$ we mean a nonsingular curve X in $\mathbb{P}^3$, of degree 4, not contained in any $\mathbb{P}^2$, and which is abstractly isomorphic to $\mathbb{P}^1$. Thus, in case $X \subseteq \mathbb{P}^3$, X is necessarily a nonsingular rational quartic curve in $\mathbb{P}^3$ (Observe that by the formula $p_a(X) = p_a(\widetilde{X}) + \sum_{P \in X} \delta_P$, $g(X) = p_a(X) = 0 \Rightarrow$ by $p_a(\widetilde{X}) \geq 0$, and $\delta_P \geq 0$ we have $\delta_P = 0$ for any $P \in X$, whence X is nonsingular). Now suppose that $X \subseteq \mathbb{P}^4$. By *rational normal quartic curve* in $\mathbb{P}^4$ we mean the $4-$uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^4$. Then the resulting variety in $\mathbb{P}^4$ is given explicitly by the parametric equations $x_0 = t^4$, $x_1 = t^3u$, $x_2 = t^2u^2$, $x_3 = tu^3$, $x_4 = u^4$. Now we will show that any nonsingular curve X in $\mathbb{P}^4$, of degree 4, not contained in any $\mathbb{P}^3$, and which is abstractly isomorphic to $\mathbb{P}^1$, can be obtained from the variety defined by the above stated parametric equations by an automorphism of $\mathbb{P}^4$. Let X be such a curve in $\mathbb{P}^4$. The embedding of X in $\mathbb{P}^4$ is determined by the linear system $\mathfrak{d}$ of hyperplane sections of X. $\mathfrak{d}$ is a linear system on X of dimension 4, because the planes in $\mathbb{P}^4$ form a linear system of dimension 4, and by hypothesis X is not contained in any $\mathbb{P}^3$, so the map $\Gamma(\mathbb{P}^4, \mathcal{O}(1)) \to \Gamma(X, i^*\mathcal{O}(1))$ is injective. Besides this, $\mathfrak{d}$ is a linear system of degree 4, since X is a curve of degree 4. By the degree of a linear system on a complete nonsingular curve, we mean the degree of any of its divisors, which is independent of the divisor chosen from this system. Now viewing X as $\mathbb{P}^1$, the linear system $\mathfrak{d}$ must correspond to a $5-$dimensional subspace $V \subseteq \Gamma(\mathbb{P}^1, \mathcal{O}(4))$. But $\Gamma(\mathbb{P}^1, \mathcal{O}(4))$ itself has dimension 5, so $V = \Gamma(\mathbb{P}^1, \mathcal{O}(4))$ and $\mathfrak{d}$ is a complete linear system. Since the embedding is determined via the linear system and the choice of basis of V, we simply conclude that X is identical to the $4-$uple embedding of $\mathbb{P}^1$, except for the choice of a basis of V. This shows that there is an automorphism of $\mathbb{P}^4$ sending the given variety by the above stated parametric equations to X. Hence $X \subseteq \mathbb{P}^4$ and not contained in any $\mathbb{P}^3$ implies that X differs from the $4-$uple embedding of $\mathbb{P}^1$ in $\mathbb{P}^4$ by an automorphism of $\mathbb{P}^4$, whence X is a rational normal quartic curve in $\mathbb{P}^4$.

(2.) Suppose that $X \subseteq \mathbb{P}^2$. Then the well-known Plücker's genus formula implies that $g = \binom{d-1}{2} = \frac{1}{2}(d-1)(d-2) = \frac{1}{2}(3)(2) = 3$.

(3.) Assume $X \subseteq \mathbb{P}^3$. Then there exists a point $O \notin X$ the projection from the point O defines a birational morphism $\varphi : X \to \varphi(X) \subseteq \mathbb{P}^2$, with at most nodes as singularities. Then the image $\varphi(X)$ cannot be nonsingular. Hence there exists at least one node, and as $r \geq 1$ we get $g(X) = g(\varphi(X)) = \frac{1}{2}(d-1)(d-2) - r = 3 - r$. Thus $g(X) < 3$, and $g \neq 0$ since that case was covered in (1.). Taking a hyperplane H and using Riemann-Roch gives $l(H) - l(K - H) = \deg H + 1 - g$. $\deg H = 4$ since X is a degree 4 curve. If $g = 2$ then $\deg K = 2g - 2 = 2$, and then $l(K - H) = 0$. Then $l(H) = 4 + 1 - 2 = 3$, but $l(H) = \dim H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) = 4$, a contradiction. Thus $g \neq 2$. Now the curve of type $(2, 2)$ has genus $g = 1$. (whose computation is explicitly depicted in the section "Nonsingular Curves on Quadric Surfaces") Since the degree of a curve of type (a, b) is $a + b$ (through the embedding and intersection with a hyperplane - (a, b) corresponds to a lines in one direction and b lines in the other). Thus curves with $g = 1$ and degree 4 exist in $\mathbb{P}^3$.

As all curves of degree 4 has been listed in the above discussion, there is no curve X of degree 4 and genus $g(X) = 2$ in $\mathbb{P}^3$, thereby proving that the curve X given by its explicit degree-4 defining polynomial is not the projection of a nonsingular curve $\widetilde{X} \subseteq \mathbb{P}^3$. Thus we conclude that although the projection of a curve $\widetilde{X} \subseteq \mathbb{P}^3$ into $\mathbb{P}^2$ results in a singular curve with singularities as nodes, the converse that *every singular curve with only nodes as singularities is a projection of a nonsingular curve in* $\mathbb{P}^3$ is **not true**.

1.9 Results to be Used Frequently

Theorem 1.9.1 *(Adjunction Formula) For a smooth (nonsingular) curve $\mathcal{C}$ on a surface X we have the adjunction formula*

$$K_C = K_X \otimes \mathcal{O}_X(\mathcal{C}) \mid_{\mathcal{C}}$$

where K_X is the canonical line bundle (and also canonical divisor) on X, and $K_{\mathcal{C}}$ is the canonical divisor on $\mathcal{C}$. The above statement is generally read in the form:

$$2g(\mathcal{C}) - 2 = \mathcal{C} \cdot (K_X + \mathcal{C}).$$

Proof: (cf. [6], pg. 13) This follows from the normal bundle sequence

$$0 \to T_{\mathcal{C}} \to T_X \mid_{\mathcal{C}} \to N_{\mathcal{C}/X} \to 0$$

which yields $\det(T_X \mid_{\mathcal{C}}) = (K_X \mid_{\mathcal{C}})^{-1} = K_{\mathcal{C}}^{-1} \otimes \mathcal{O}_X(\mathcal{C}) \mid_{\mathcal{C}}$. Thus, if $g(\mathcal{C}) = g$ is the genus of $\mathcal{C}$, then

$$2g - 2 \;=\; \deg\, K_{\mathcal{C}} \;=\; (K_X + \mathcal{C}) \cdot \mathcal{C}.$$

□

Theorem 1.9.2 *(Hodge Index Theorem): Let H be an ample divisor on a variety X, and let D be a divisor such that $D \cdot H = 0$. Then $D^2 \leq 0$, and if $D^2 = 0$, then $D \cdot E = 0$ for all divisors E.*

Theorem 1.9.3 *(Riemann inequality) Let X be a nonsingular projective curve in some $\mathbb{P}^n$. Then there exists a constant $\gamma = \gamma(X)$ such that for every divisor D on X the inequality*

$$l(D) \geq \deg D - \gamma$$

holds.

Theorem 1.9.4 *(Riemann-Roch Version 1) Let D be an arbitrary divisor on a curve, K the canonical class, and g the genus of the curve. Then the following equality holds:*

$$l(D) - l(K - D) = 1 - g + \deg\, D.$$

Theorem 1.9.5 *(Riemann-Roch Version 2) For a divisor D on a surface X, we have the* `Riemann-Roch` *equality*

$$l(D) - s(D) + l(K - D) = \frac{1}{2} D \cdot (D + K) + 1 + p_a$$

where $s(D)$ is the least degree of the surface containing D.

Theorem 1.9.6 (Hurwitz's Theorem) *For a finite separable morphism of curves* $f : X \to Y$*, with* $n = \deg\ f$ *we have the equality*

$$2g(X) - 2 = n \cdot (2g(Y) - 2) + \deg R.$$

where $R = \sum_{P \in X} \text{lenght}(\Omega_{X/Y})_P \cdot P$*. Furthermore, if f has only tame ramification, then*

$$\deg\ R = \sum_{P \in X} (e_P - 1).$$

Theorem 1.9.7 (Bertini's Theorem) *Let* X *be a nonsingular closed subvariety of* $\mathbb{P}_k^n$ *defined over an algebraically closed field* k*. Then there exists a hyperplane* $H \subseteq \mathbb{P}_k^n$*, not containing* X*, and such that the scheme* $H \cap X$ *is regular at every point. Furthermore, if* $\dim\ X \geq 2$*, then* $H \cap X$ *is connected, hence irreducible, and therefore* $H \cap X$ *is a nonsingular variety. Moreover, the set of hyperplanes satisfying this condition forms an open dense subset of the complete linear system* $|H|$*, considered as a projective space.*

1.10 Statement of the Problem

Classification of objects studied in any part of mathematics is a problem commonly treated , e.g. classification of groups, fields, differential equations, $\mathcal{C}^\infty$ manifolds, covering maps,. . . ,etc. As algebraic sets are the main objects of interest in algebraic geometry, classification of algebraic sets, and/or varieties is a problem at the heart of the field. This problem can be seperated into a class of problems with respect to the dimension of varieties concerned, i.e. classification of curves, surfaces, 3-folds, . . . , etc. As curves have the least nontrivial dimension, they have been naturally considered as a first step towards the classification of algebraic varieties and yet it has proven no ease to deal with.

The most general notion that can be granted for an algebraic curve is that of an *abstract algebraic curve*, as we have defined it. We have seen that every abstract algebraic curve can be embedded into a projective space $\mathbb{P}^n$, then considering only projective curves gives us the most general setting. In any trial to classify projective varieties, there seems two possible routes: either up to isomorphism or up to birational equivalence, the former having a global perspective and the latter a local one. In case there is a dominant rational map between two nonsingular curves, then there is also a dominant morphism between them. Hence any two birationally equivalent nonsingular projective curves are necessarily isomorphic. Therefore the two different routes appearing toward the classification problem coincides for nonsingular curves in $\mathbb{P}^n$. Any nonsingular algebraic curve in $\mathbb{P}^n$ can be mapped to $\mathbb{P}^3$ by successive projections so that the resulting image is still nonsingular, this kind of a projection is a a birational equivalence which preserve the genus of the original curve in $\mathbb{P}^n$. Hence classification of all nonsingular curves in $\mathbb{P}^3$ includes classification of all abstract nonsingular curves up to birational equivalence, so builds the most general setting.

In the literature, the first notable attempt of classification of nonsingular projective curves in $\mathbb{P}^3$ has been started by Halphen [11] and Max Noether [28] in 1882, providing a classification of curves up to degree 20. In their trial, Halphen and Noether have described methods of constructions for curves subject to certain conditions. Concerning nonsingular projective curves, we can talk about two discrete invariants: degree and genus. By these two integers, we can divide the class of nonsingular projective curves into sub-classes which can substitute a basis for the classification problem. Although *genus* is a birational invariant, *degree* depends on the projective embedding of abstract curves; so not being a unique quantity related to the intrinsic structure of the abstract curve (e.g. the twisted cubic has degree 3 in $\mathbb{P}^3$, although it is isomorphic to the projective line $\mathbb{P}^1$).

The main problems connected with the classification of nonsingular algebraic curves can be listed as follows:

Problem-1: Is it possible that for every pair of nonnegative integers

$(d, g) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ there exists an irreducible non-singular algebraic curve Y of degree d and genus g in $\mathbb{P}^3$. If it is not possible, for what kind of pairs (d, g) there exists an irreducible non-singular curve Y of degree d and genus g, which lies in $\mathbb{P}^3$?

Problem-2: For admissible pairs of nonnegative integers (d, g) describe the set of all curves Y of degree d and genus g in $\mathbb{P}^3$ depicting its irreducible components and the dimension of the parameter space which parametrizes the algebraic families of all such curves with degree-genus pair (d, g).

In case **Problem-2** has a solution; For each such degree-genus pair (d, g) let $H_{d,g}$ denote the set of all irreducible nonsingular curves in $\mathbb{P}^3$ with genus g and degree d. This set $H_{d,g}$ deserves examining its irreducible components, dimension, singular points,... once an algebraic structure has been introduced on it.

Problem-3: Describe the algebro-geometric properties of each curve $\mathcal{C}$ of degree d and genus g: least degree of a surface containing it, postulation (for a detailed description of this notion, see [15]), normal bundle, existence of special linear series, etc.

Problem-4: What is the largest integer e for which $H^1(\mathcal{C}, \mathcal{O}_c(e)) \neq 0$?

In his attempt to classify algebraic space curves, Halphen has given the following conjecture in the form of a theorem with a wrong proof which lied on general position arguments that later was understood to be wrong.

Conjecture (Halphen) : All possible values of the degree-genus pairs (d,g) of irreducible nonsingular curves Y in $\mathbb{P}^3$ are determined as follows;

(a) Plane curves, for any $d > 0$, with

$$g = \frac{1}{2}(d-1)(d-2).$$

(b) Curves on quadric surfaces, for any $a, b > 0$, with

$$d = a + b,$$

$$g = (a-1)(b-1).$$

(c) If Y does not lie on a plane or a quadric surface, then

$$g \leq \frac{1}{6}d(d-3) \ + \ 1.$$

(d) For any given $d > 0$, and g with $0 \leq g \leq \frac{1}{6}d(d-3) \ + \ 1$, there is a curve $Y \subseteq \mathbb{P}^3$ with degree d, and genus g.

Halphen claimed to construct curves with degree-genus pairs (d, g), for all $d \in \mathbb{Z}^+$ and g within the bounds stated in (c), on cubic surfaces. It has later been understood that construction of all such curves is not possible even on a *singular cubic surface*, that there are some gaps, some values of the pair (d, g) which falls into the interval stated in part (c) of Halphen's theorem for which there is no such curve on a cubic surface, not even on a singular cubic surface. We see that as the dimension of the surface on which a curve $\mathcal{C}$ lies increases, then the corresponding genus $g(\mathcal{C})$ of the curve exhibits a decline. Hence the greatest possible value of the genus of a curve $\mathcal{C}$ with a fixed degree d is $g = \binom{d-1}{2} = \frac{1}{2}(d-1)(d-2)$. Since a curve with any degree exists simply by the d-uple embedding of $\mathbb{P}^1$ into a projective space $\mathbb{P}^n$, since Halphen and Noether's trial towards the classification problem mathematical research has concentrated on the existence of curves on suitable surfaces on which the curve lies. In connection with this viewpoint the following invariants have been defined (cf. [15]).

$d =$ degree of $\mathcal{C}$,
$g =$ genus of $\mathcal{C}$,
$s =$ least degree of a surface containing $\mathcal{C}$,
$e =$ least integer for which $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(e)) \neq 0$.

One could ask what possible 4−uples (d, g, s, e) occur for curves in $\mathbb{P}^3$, but this is a more difficult problem than finding only the admissible degree-genus pairs (d, g).

The complete answer to Problem-1 has been completed by Grueson, Peskin and Mori in 1982-1984. Hence, by now it is a problem with a complete answer, which also led to other problems on which mathematical research focuses.

Trials to answer Problem-2 in an attempt to find the parameter space have created the concepts of *Chow variety*, *Hilbert scheme*, and *moduli space*, which are a finite union of quasi-projective varieties, defined in a natural way. Although Problem-1 has been answered completely, Problem-2 still needs a complete answer and therefore a subject treated in research.

The Hilbert scheme parametrizes all 1-dimensional closed subschemes of $\mathbb{P}^3$ with a given degree d and arithmetic genus p_a. The Chow variety is a parameter space consisting of cycles of a specific dimension r, and degree d in $\mathbb{P}^n$ modulo rational equivalence. As a set $\mathcal{M}_g$ is the set of equivalence classes of nonsingular curves of genus g modulo isomorphism, which was first introduced by David Mumford as abstract $(3g-3)$-dimensional varieties, later shown to be irreducible quasi-projective varieties by Mumford and Deligne. As a variety $\mathcal{M}_g$ is irreducible and $\mathcal{M}_g$ is not a projective variety, but its closure $\overline{\mathcal{M}}_g$ is. The points of $\overline{\mathcal{M}}_g$ not in $\mathcal{M}_g$ are called stable curves.

The results culminated in efforts trying to classify the parameter spaces of algebraic curves corresponding to certain data prescribed on such curves can be summarized as it is depicted in the following table (cf. [31], pg. 71).

Starting point	Underlying Set	Parameter Space
A curve $\mathcal{C}$	Pic $^0(\mathcal{C}) = \{$ divisors of degree 0 modulo linear equivalence $\}$ Not only a set, but also a functor **Sch** $\to$ **Set**	Closed points of Jacobian variety J. J itself represents the functor.
$\mathbb{P}^n_k$ and $P(z) \in \mathbb{Q}[z]$	set of closed subschemes $Z \subseteq \mathbb{P}^n_k$ with associated Hilbert polynomial $P(z)$	The Hilbert scheme $Hilb(\mathbb{P}^n)$
fix $g > 0$	isomorphism classes of curves of genus g	The variety $\mathcal{M}_g$ of moduli. Fine if represents the functor, coarse if not.
$\mathbb{P}^n_k$, degree d, and dimension r	set of cycles of dimension r and degree d in $\mathbb{P}^n$ modulo rational equivalence.	The Chow scheme (or Chow variety). It does not represent a functor.

Chapter 2

Solution to the Classification Problem

2.1 Trivial Cases - Low Genus

2.1.1 Genus 0 Curves

As predicted, genus 0 curves are rather easy to classify no matter what their projective embedding is. Hence in what follows we merely assume that $\mathcal{C}$ is a curve lying in any $\mathbb{P}^n$ with genus $g(\mathcal{C}) = 0$.

Let $\mathcal{C}$ be a curve of genus $g = 0$ and let $P \in \mathcal{C}$ be a closed point. There is an exact sequence

$$0 \to \mathcal{O}_{\mathcal{C}} \to \mathcal{L}(P) \to \kappa(P) \to 0.$$

As a divisor P corresponds to the invertible sheaf $\mathcal{L}(P)$. Let $s \in \Gamma(\mathcal{L}(P))$ be a global section which generates $\mathcal{L}(P)$ as an $\mathcal{O}_{\mathcal{C}}$-module. Thus s has a pole of order

1 at P and no other poles. Then s defines a morphism

$$\mathcal{O}_{\mathcal{C}} \to \mathcal{L}(P) \quad : \quad 1 \mapsto s$$

which has cokernel $\kappa(P)$. Since $g(\mathcal{C}) = 0 = \dim_k H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$, we must have $H^1(\mathcal{O}_{\mathcal{C}}) = 0$. Thus taking cohomology yields

$$0 \to H^0(\mathcal{O}_{\mathcal{C}}) \to H^0(\mathcal{L}(P)) \to H^0(\kappa(P)) \to H^1(\mathcal{O}_{\mathcal{C}}) = 0.$$

Since $H^0(\mathcal{O}_{\mathcal{C}}) = k$ and $H^0(\kappa(P)) = k$ it follows that $H^0(\mathcal{L}(P)) = k \oplus k$. View $\mathcal{O}_{\mathcal{C}} \subseteq \mathcal{L}(P) \subseteq K$ where $K = K(X)$ is the function field sheaf. Then

$$H^0(\mathcal{O}_{\mathcal{C}}) \to H^0(\mathcal{L}(P)) \to K.$$

Since $\dim H^0(\mathcal{L}(P)) > \dim H^0(\mathcal{O}_{\mathcal{C}})$, there is $f \in H^0(\mathcal{L}(P)) \backslash H^0(\mathcal{O}_{\mathcal{C}})$ and f is non constant. Thus $f \in K = K(X)$ and $f \notin k$. So f defines a morphism $C \to \mathbb{P}^1$ as follows. If $Q \in \mathcal{C}$ and $f \in \mathcal{O}_Q$ send Q to $\overline{f} \in \mathcal{O}_Q/\mathfrak{m}_Q = k \subseteq \mathbb{P}^1$. If $Q \notin \mathcal{C}$ so that f has a pole at Q, send Q to $\infty \in \mathbb{P}^1$ (Here we identify k with $[k : 1]$, and ∞ with the point $[1 : 0]$). Since f lies in $H^0(\mathcal{L}(P))$ which has dimension 1 over $k = H^0(\mathcal{O}_{\mathcal{C}})$, $v_Q(f) = -1$. Thus ∞ does not ramify so the degree of the morphism defined by f is the number of points lying over ∞ which is 1. Therefore $\mathbb{P}^1 \cong \mathcal{C}$.

2.1.2 Genus 1 Curves

Before further mentioning the classification of genus 1 curves in $\mathbb{P}^3$, we present the following theorem which is a nice characterization of nonsingular cubics sitting in $\mathbb{P}^2$.

Lemma 2.1.1 *Let $X \subseteq \mathbb{P}^2$ be a nonsingular cubic; then*

$$l(D) = degD \quad \text{for every effective divisor } D > 0 \text{ on } X.$$

Conversely, a curve for which the above condition holds is isomorphic to a non-singular cubic.

Proof: Let $X \subseteq \mathbb{P}^2$ be a nonsingular cubic. By Plücker's genus formula $g(X) = \binom{2}{2} = \frac{1}{2}(3-1)(3-2) = 1$. Then by Riemann-Roch theorem for any divisor D on X we have $l(D) - l(K-D) = 1 - g + \deg\ D = \deg\ D$, and since $l(K-D) = \dim_k \mathcal{L}(K-D) \geq 0$ we have the trivial inequality $l(D) \leq \deg D$, and hence it is enough to prove that $l(D) \geq \deg D$. Suppose D is any effective divisor. Pick any point α_0 of the nonsingular cubic curve X. We show that there exists a point $\alpha \in X$ such that for some $k \in \mathbb{Z}$

$$D \sim \alpha + k\alpha_0$$

If $degD = 1$, then for $k = 0$ our required equality holds. If $degD > 1$ then $D = D' + \beta$ with $degD' = degD - 1$ and $D' > 0$. Using induction, we can assume that the required equality above is proved for D', that is, $D' \sim \gamma + l\alpha_0$. Then $D \sim \beta + \gamma + l\alpha_0$. If we can find a point α such that

$$\beta + \gamma \sim \alpha + \alpha_0,$$

then our required equality for D will follow. Suppose first that $\beta \neq \gamma$. Pass the line given by $L = 0$ through β and γ. By Bézout's theorem, $L \cdot X = 3$, hence

$$divL = \beta + \gamma + \delta \quad \text{for some } \delta \in X.$$

Suppose moreover that $\delta \neq \alpha_0$ and pass the line given by $L_1 = 0$ through δ and α_0. In the same way as for the last equality we get $divL_1 = \delta + \alpha_0 + \alpha$ for some $\alpha \in X$. Since $divL \sim divL_1$ we get $\beta+\gamma+\delta \sim \delta+\alpha_0+\alpha$ and hence $\beta+\alpha \sim \alpha+\alpha_0$, therefore $D \sim \alpha + k\alpha_0$.

So, for any effective divisor D on X and any chosen point $\alpha_0 \in X$ there exists a point $\alpha \in X$ such that $D \sim \alpha + k\alpha_0$, for some $k \in \mathbb{Z}^+ \cup \{0\}$. Then $\mathcal{L}(D) \cong \mathcal{L}(\alpha + k\alpha_0)$ and $\mathcal{L}(K - D) \cong \mathcal{L}(K - \alpha - k\alpha_0) \Rightarrow$ by Riemann-Roch theorem $\deg\ D = \deg\ (\alpha + k\alpha_0) = 1 + k$. Hence $l(\alpha + k\alpha_0) > k$, together with $l(\alpha+k\alpha_0) \leq \deg\ (\alpha+k\alpha_0) = 1+k$ implies that $k < l(\alpha+k\alpha_0) = l(D) \leq 1+k \Rightarrow l(D) = k+1 = \deg\ D$. Thus to prove the assertion of the theorem it is enough to prove that the strict inequality $l(\alpha+k\alpha_0) > k$ holds. If $k = 1$, then $l(\alpha+\alpha_0) > 1$; because $\mathcal{L}(\alpha + \alpha_0)$ contains nonconstant functions L_1/L_0, where L_0 is defining equation of the line through the points α and α_0 and L_1 any line through the third point of the intersection L_0 and X.

Hence for $k > 1$ it suffices to exhibit a function f_k with $(f_k)_\infty = k\alpha_0$; indeed, then $f_k \in \mathcal{L}(k\alpha_0) \subseteq \mathcal{L}(\alpha + k\alpha_0)$ and $f_k \notin \mathcal{L}(\alpha + (k-1)\alpha_0)$, whence $l(\alpha + k\alpha_0) \geq l(\alpha + (k-1)\alpha_0) + 1$, and our assertion is proved by induction. It is easy to find such f_k with the required property for $k = 2$ or $k = 3$. Namely, $f_2 = \frac{L_1}{L_0}$, where L_0 is the tangent line to X at the point α_0, and L_1 is any line through the third point of the intersection of L_0 and X. In the same fashion, $f_3 = \frac{L_1 L_3}{L_0 L_2}$, where L_0 and L_1 are as before, L_2 is the defining equation of the line through α_0 and one of the other points of intersection of L_1 and X, and $L_3 = 0$ a line through the third point of intersection of L_2 and X. Finally, if $k = 2r$ is even, then $f_k = f_2^r$; and if $k = 2r + 3$ is odd and ≥ 3 then $f_k = f_3 f_2^r$. Hence the required equality $l(D) = degD$ holds for any effective divisor $D > 0$ on X (cf. [29], pg. 177).

Conversely, suppose that X is a nonsingular projective curve such that for any effective divisor $D > 0$, $l(D) = degD$. Take any arbitrary point $p \in X$. Since $\mathcal{L}(2p) > 1$, there exists a function $t \in k(X)$ with $div_\infty(t) = 2p$ (note that since the eqaulity $div_\infty(t) = p$ forces X to be rational, it is not attained). By our equality $\mathcal{L}(3p) \neq \mathcal{L}(2p)$, so that there exists a function $u \in k(X)$ with $div_\infty(u) = 3p$. Finally, $\mathcal{L}(6p) = 6$. But we already know 7 functions belonging to $\mathcal{L}(6p)$, namely $1, t, t^2, t^3, u, tu, u^2$. Hence there must be a linear dependence relation between these

$$a_0 + a_1 t + a_2 t^2 + a_3 t^3 + b_0 u + b_1 tu + b_2 u^2 = 0.$$

Thus the functions t and u define a rational, hence a regular map f from X to the plane cubic $Y \subseteq \mathbb{P}^2$ with the above equation in inhomogeneous coordinates. This is the rational map defined by the linear system $\mathcal{L}(3p)$.

The map f defines an inclusion of function fields $f^* : k(Y) \hookrightarrow k(X)$. Let us prove that $f^*(k(Y)) = k(X)$. For this, remark that $k(Y) \supseteq k(t)$ and $k(Y) \supseteq$

$k(u)$, and the functions t and u each defines a map of $X \to \mathbb{P}^1$. By assumption $div_\infty(t) = 2p$, which means that the map g defined by t satisfies $g^*(\infty) = 2p$. Then, since $deg\ f = deg(f^*(y))$ for any point $y \in Y$, it follows that $deg\ f = 2$, that is $[k(X) : k(f^*(t))] = 2$. Similarly, $[k(X) : k(f^*(u))] = 3$. Since $[k(X) : f^*(k(Y))]$ has to divide both of these numbers, $k(X) = f^*(k(Y))$, that is, f is birational. The cubic given by the above equation in t,and u cannot have singular points, since then it, and X together with it, would be a rational curve, which contradicts the equality $l(D) = deg\ D$ assertion of the theorem. Therefore Y is nonsingular cubic, and hence f is an isomorphism. □

And now our theorem important for the classification of curves with genus 1 follows:

Theorem 2.1.1 *If a curve $\mathcal{C}$ has genus $g(\mathcal{C}) = 1$ then $\mathcal{C}$ is isomorphic to a cubic in $\mathbb{P}^2$.*

Proof: Start by observing that if $\deg D > 2g-2$ for a divisor D on a nonsingular projective curve X then $l(D) = 1-g+\deg D$. To see this, suppose that $\deg D > 2g-2$ then by Riemann-Roch theorem we have $l(D) - l(K-D) = 1-g+\deg D$, and (writing the same equality for $K-D$ this time) $l(K-D) - l(K-(K-D)) = 1-g+\deg(K-D) \Rightarrow l(K-D)-l(D) = 1-g+\deg(K-D) \Rightarrow -(l(D)-l(K-D)) = 1-g+\deg(K-D) \Rightarrow g-1-\deg D = 1-g+\deg(K-D) \Rightarrow \deg(K-D) = 2g-2-\deg D < 2g-2+2-2g = 0$, so $\deg(K-D) < 0$. But then $l(K-D) = 0$. But then as we have $l(D) - l(K-D) = l(D)$, by Riemann-Roch theorem we obtain $l(D) = 1-g+\deg D$. For $g = 1$, since for any effective divisor $D > 0$ on X we have automatically $\deg D > 0 = 2g-2$ we must have by the above notified observation $l(D) = 1-g+\deg D = \deg D$. But then by the previous lemma, X is isomorphic to a nonsingular cubic in $\mathbb{P}^2$. □

We have seen that an elliptic curve $\mathcal{C}$ in $\mathbb{P}^2$ can be represented in inhomogeneous coordinates by an equation of the form

$$ay^2 + bx^3 + cxy + dx^2 + ey + fx + h = 0.$$

for some $a, b, c, d, e, f, h \in k$. Furthermore, both x^3 and y^2 occur with a nonzero coefficient, since they are the only functions with a 6−fold pole at p (where p was the chosen point on our curve with $div_\infty(y) = 3p$). Thus replacing y by a suitable scalar multiple we may assume $a = 1$. Preparing to complete the square we may rewrite the above relation as

$$y^2 + (cxy + ey) + (\frac{1}{2}cx + \frac{1}{2}e)^2 - (\frac{1}{2}cx + \frac{1}{2}e)^2 + bx^3 + dx^2 + fx + h = 0$$

Replacing y by $\frac{1}{2}cx + \frac{1}{2}e$ transforms the equation into

$$y^2 = e(x-a)(x-b)(x-c).$$

where $a, b, c, e \in k$ are new appropriate constants. Next absorbing e (by a suitable coordinate exchange if necessary) yields

$$y^2 = (x-a)(x-b)(x-c).$$

At this stage, let us translate x by a to obtain

$$y^2 = x(x-a)(x-b).$$

Multiplying and dividing by a^3 yields

$$y^2 = a^3\frac{x}{a}(\frac{x}{a} - 1)(\frac{x}{a} - \frac{b}{a}).$$

Replacing x by $\frac{x}{a}$ and absorbing a^3 we obtain

$$y^2 = x(x-1)(x-\lambda)$$

where $\lambda \neq 0, 1$ by the nonsingularity of $\mathcal{C}$. Observe that while doing all the operations above, we only need the hypothesis char $k \neq 2$, and $\overline{k} = k$ on the field k.

Elliptic curves are classified by their j-invariants, which is defined for an elliptic curve with affine equation $y^2 = x(x-1)(x-\lambda)$ as

$$j(\lambda) = 2^8\frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(1-\lambda)^2}.$$

And two elliptic curves $\mathcal{C}_1$ and $\mathcal{C}_2$ are isomorphic if and only if $j(\mathcal{C}_1) = j(\mathcal{C}_2)$. Moreover the moduli space $\mathcal{M}_1$ is isomorphic to 1−dimensional affine space $\mathbb{A}^1$, where each point in $\mathbb{A}^1$ viewed setwise being an element of k is attained by an elliptic curve $\mathcal{C} \subseteq \mathbb{P}^2$ as its j−invariant $j(\mathcal{C})$ (For the details and properties of the j−invariant cf. [13] pp. 317-320).

2.1.3 Genus 2 Curves

Suppose that $\mathcal{C} \subseteq \mathbb{P}^3$ is a nonsingular curve with genus $g(\mathcal{C}) = 2$, over a field k with char $k \neq 2$. Let $\Omega_{\mathcal{C}}$ be the $k-$vector space of everywhere regular differential forms on $\mathcal{C}$, and $K_{\mathcal{C}}$ the canonical divisor class of $\mathcal{C}$, which is the linear equivalence class of differential $1-$forms on $\mathcal{C}$. Then dim $\Omega_{\mathcal{C}} = 2$ and deg $K_{\mathcal{C}} = 2g - 2 = 2$. If ω_1, ω_2 constitute a basis of $\Omega_{\mathcal{C}}$, then $f = \omega_1/\omega_2$ defines a finite morphism $f : \mathcal{C} \to \mathbb{P}^1$ (It might happen that (ω_1) and (ω_2) have a common zero x_0, in which case $f^{-1}(0) = (\omega_1) - x_0$, and $f^{-1}(\infty) = (\omega_2) - x_0$. But then f would have degree 1 and hence be birational to $\mathbb{P}^1$, in which case $\mathcal{C}$ would have genus 0 not 2). But then by Hurwitz's theorem we get $2g(\mathcal{C}) - 2 = 2 = 2(0 - 2) + \deg R$ implying that deg $R = 6$. Since deg $f = 2$, each ramification index $e_P \leq 2$, hence each point P in the support of the ramification divisor R occurs with degree $1 \leq e_P - 1 \leq 2 - 1 = 1$. Thus the morphism f is ramified at exactly 6 points, say $\alpha_1, \ldots, \alpha_6 \in \mathcal{C}$, with ramification index 2 at each of $\alpha_i \in \mathcal{C}$.

Conversely, given six different elements $\alpha_1, \ldots, \alpha_6 \in k$, let K be the extension of $k(x)$ by the equation $z^2 = (x - \alpha_1) \cdots (x - \alpha_6)$. Let $f : X \to \mathbb{P}^1$ be the corresponding morphism of curves, where X is defined by the affine equation $z^2 = (x-\alpha_1) \cdots (x-\alpha_n)$. Now observe that for a square-free nonconstant polynomial $F \in k[x_1, \ldots, x_n]$, $A = k[x_1, \ldots, x_n, z]/(z^2 - F)$ is an integrally closed ring. To see this result, note that the quotient field K' of A is just $k(x_1, \ldots, x_n)[z]/(z^2 - F)$, and this quotient field is a Galois extension of $k(x_1, \ldots, x_n)$ with Galois group $\mathbb{Z}/2\mathbb{Z}$ generated by $z \mapsto -z$. If $\alpha = g + hz \in K'$, where $g, h \in k(x_1, \ldots, x_n)$, then the minimal polynomial of α is $X^2 - 2gX + (g^2 - h^2F)$. Hence α is integral over $k[x_1, \ldots, x_n]$ if and only if $g, h \in k[x_1, \ldots, x_n]$. Hence A is the integral closure of $k[x_1, \ldots, x_n]$ in K'. Therefore the ring $k[x, z]/(z^2 - (x - \alpha_1) \cdots (x - \alpha_6))$, which is the coordinate ring $A(X)$ of X, is integrally closed, whence X is a normal variety. Moreover $k(X)$ is a quadratic extension of $k(\mathbb{P}^1)$, hence if t is a coordinate on $\mathbb{P}^1$ we must have $k(X) = k(t, \sqrt{Q(t)})$ for some polynomial Q. Now observe that the branch points of f are precisely the points of X over the zeroes of Q of odd order, where $Q(t) = (t - \alpha_1) \cdots (t - \alpha_6)$. Hence f is ramified only at the points $x = \alpha_i \in \mathbb{P}^1$, with ramification index 2 at each point. Then deg $R = \sum_{i=1}^{6}(e_{\alpha_i} - 1) = 6$. But then since $g(\mathbb{P}^1) = 0$ by Hurwitz's theorem we get $2g(X) - 2 = 2.(0-2) + 6 = 2 \Leftrightarrow g(X) = 2$, and f is the same map as the one determined by the canonical system.

We can consider $\mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\} = [k:1] \cup [1:0]$. Defining a *fractional linear transformation* of $\mathbb{P}^1$ by sending $x \mapsto \frac{ax+b}{cx+d}$, for $a, b, c, d \in k$ with $ad - bc \neq 0$, we observe that a fractional linear transformation induces an automorphism of $\mathbb{P}^1$, which we denote by $\mathbb{PGL}(1,k) = \mathbb{GL}(2,k)/k^\times$. Also we can observe that $\mathrm{Aut}\mathbb{P}^1 \cong \mathrm{Aut}k(x)$, and that every automorphism of $k(x)$ is a fractional linear transformation, and hence $\mathrm{Aut}\mathbb{P}^1 \cong \mathbb{PGL}(1,k)$. Now if P_1, P_2, P_3 are 3 distinct points of $\mathbb{P}^1$, there exists a unique $\varphi \in \mathrm{Aut}\mathbb{P}^1$ such that $\varphi(P_1) = 0$, $\varphi(P_2) = 1$, and $\varphi(P_3) = \infty$ (Such φ can be found explicitly by solving the matrix equation $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} P_i \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_i \\ y_i \end{smallmatrix}\right)$). Thus, since we have shown that a nonsingular curve $\mathcal{C}$ is ramified at exactly 6 points $\alpha_1, \ldots, \alpha_6 \in \mathbb{P}^1$, we may order the 6 ramified points $x = \alpha_i \in \mathbb{P}^1$, and then normalize by sending the first three $0, 1$, and ∞, respectively; and hence we may assume that $\mathcal{C}$ is ramified over the points $0, 1, \infty, \beta_1, \beta_2, \beta_3$, where $\beta_1, \beta_2, \beta_3$ are three distinct elements of k different from 0 and 1.

Let Σ_6 be the symmetric group of 6 letters. Define an action of Σ_6 on the sets of three distinct elements of $k \backslash \{0,1\}$ as follows: reorder the set $0, 1, \infty, \beta_1, \beta_2, \beta_3$ according to a given element of $\sigma \in \Sigma_6$, then normalize as explained in the previous paragraph so that the first three becomes $0, 1, \infty$ again. Then the last three are new $\beta_1', \beta_2', \beta_3'$.

Summing up what we have done above, we simply conclude that there is a one-to-one correspondence between the set of isomorphism classes of $genus - 2$ curves over k, and triples of distinct elements $\beta_1, \beta_2, \beta_3 \in k \backslash \{0,1\}$ modulo the action of Σ_6 described in the preceding paragraph. In particular, there are many non-isomorphic curves of genus 2. We simply state that curves of genus 2 depend only on 3 parameters, since they correspond to the points of an open subset of

$\mathbb{A}_k^1$ modulo a finite group.

In general, curves X of genus ≥ 2 which admit a degree 2 map

$$\pi : X \to \mathbb{P}^1$$

or, equivalently curves X having the property that $k(X)$ is a quadratic extension of a purely transcendental subfield $k(t)$ are called *hyperelliptic* curves.

2.1.4 Existence of Linear Systems of Divisors on Curves

As the genus g increases, classification problem for curves of genus g gets more complicated. In this case, examining base point free linear systems of divisors on curves opens a new road towards classification. Conventionally, it is a classical notation that a base point free linear system of degree d and dimension r is called a g_d^r. Hence, for example to say that a curve is hyperelliptic (possesses a base point free linear system of degree 2 and dimension 1 with genus $g \geq 2$) is equivalent to say that it has a g_2^1. A natural question is to ask for the least d for which there exists a g_d^1 on the curve $\mathcal{C}$. To give a rough idea (cf. [31] pp. 54-58):

- $g = 0$ there is a g_1^1 coming from the embedding $\mathbb{P}^1 \to \mathbb{P}^1$,
- $g = 1$ there are infinitely many g_2^1's,
- $g = 2$ there is a g_2^1 namely $\omega_{\mathcal{C}}$,
- $g \geq 2$ if there is a g_2^1 then $\mathcal{C}$ is hyperelliptic,
- $g > 2$ there exists nonhyperelliptic curves.

In case a curve $\mathcal{C}$ has a g_3^1 it is called trigonal. Existence of such linear systems of divisors on curves helps to distinguish between nonisomorphic curves, assisting to understand the isomorphism classes in a better way. A direct attempt to classify curves of genus $g \geq 2$ does not turn out to be as easy as the case with $g = 0, 1$, and 2. To give an example all of the following are curves of genus 6, which are mutually disjoint:

(a) degree 5 curves in $\mathbb{P}^2$,

(b) a type $(2, 7)$ curve (of degree 9) on the quadric surface Q in $\mathbb{P}^3$,

(c) a type $(3, 4)$ curve (of degree 7) on the quadric surface Q in $\mathbb{P}^3$.

For a detailed study of special linear systems of divisors on curves, see [19].

2.2 Canonical Curves

The curves other than the hyperelliptic ones admit an embedding into $\mathbb{P}^{g-1}$ called the *canonical embedding* determined by the *canonical linear system*. The resulting images in $\mathbb{P}^{g-1}$ are called *canonical curves*. Below we summarize this certain property (cf. [23] pp. 148-149).

Lemma 2.2.1 *Let X be a nonsingular nonhyperelliptic curve of genus $g \geq 3$. Then $|K_X|$ has no base points, and for all $x \in X$, $|K_X - x|$ has no base points.*

Proof: Suppose by way of contradiction that $|K_X|$ has a base point x. Then for every $\omega \in \Omega_X$ vanishes at x, hence $\Omega_X = \Omega_X(-x)$. But dim $\Omega_X = g$ and by Riemann-Roch

$$\dim \mathcal{L}(x) - \dim \Omega_X(-x) = 1 - g + 1 = 2 - g.$$

So if $\Omega_X = \Omega_X(-x)$, we find that dim $\mathcal{L}(x) = 2$, i.e., besides constants there are other functions $f \in k(X)$ with only a simple pole at x. Such an f defines a map $\pi : X \to \mathbb{P}^1$ such that $\pi^{-1}(\infty) = x$, hence deg $\pi = 1$, hence X is birational to

$\mathbb{P}^1$, hence it has genus $g = 0$, contradicting our hypothesis. Hence we must have that $|K_X|$ has no base points.

Similarly, if $|K_X - x|$ has a base point y, then $\Omega_X(-x) = \Omega_X(-x-y)$. We have just seen that dim $\Omega_X(-x) = g-1$, and by Riemann-Roch

$$\dim \mathcal{L}(x+y) - \dim \Omega_X(-x-y) = 2 - g + 1 = 3 - g.$$

So if $\Omega_X(-x) = \Omega_X(-x-y)$, we find dim $\mathcal{L}(x+y) = 2$. As above there is now a degree 2 map $\pi : X \to \mathbb{P}^1$ and this means that X is hyperelliptic: also contradicting our hypothesis. Hence $|K_X - x|$ has no points for all $x \in X$. □

Now we give the characterization of canonical curves

Proposition 2.2.1 *If X is a nonsingular nonhyperelliptic curve of genus $g \geq 3$, then the linear system $|K_X|$ of canonical divisors (ω) defines a regular map*

$$Z : X \to \mathbb{P}^{g-1}$$

which is an isomorphism between X and a nonsingular curve $X' = Z(X) \subseteq \mathbb{P}^{g-1}$ of degree $2g-2$ whose hyperplane sections are precisely the canonical divisors (ω), $\omega \in \Omega_X$.

Proof: $|K_X|$ defines a map $Z : X \to \mathbb{P}^{g-1}$ such that the divisors $Z^{-1}[H]$ are equal to the divisors (ω), $\omega \in \Omega_X$, where $[H]$ is a hyperplane in $\mathbb{P}^{g-1}$. By the lemma, for all $x, y \in X$, $x \neq y$, we can find a divisor $D \in |K_X - x|$ with $y \notin$ Supp D. Then $D+x = (\omega)$, where ω is zero at x but not at y. If $(\omega) = Z^{-1}[H]$, it follows that $Z(x) \in H$ but $Z(y) \notin H$, hence Z is injective. Thus if $X' = Z(X)$,

$$\deg X' = \deg (X' \cdot H) = \deg Z^{-1}[H] = \deg (\omega) = 2g-2.$$

Eventually, by the lemma, for all x, there is a divisor $D \in |K_X - x|$ with $x \notin$ Supp D. Then $D + x = (\omega)$ where ω has a simple zero at x. Therefore (ω) has $2g-3$ zeroes in addition to x. If $(\omega) = Z^{-1}[H]$, it follows that H meets X' at $Z(x)$ and at $2g-3$ further points. Thus $\text{mult}_x Z(x) = 1$. This proves that X' is nonsingular, hence Z is an isomorphism between X and X'. □

2.3 Complete Intersection Case

We start by treating a rather trivial case in which a curve Y is a complete intersection of two surfaces S_1 and S_2 of respective degrees a and b. Recall that a variety $X \subseteq \mathbb{P}^n$ of dimension r is a (strict) complete intersection in case its defining ideal $I(X)$ can be generated by $n - r$ elements, and it is called a *set-theoretic complete intersection* in case X can be written as the intersection of $n - r$ hypersurfaces. In case a curve Y is a complete intersection, (in which case it is the intersection of two surfaces in $\mathbb{P}^3$) its degree and genus are rather easy to compute as it is depicted in the following proposition.

Proposition 2.3.1 *Let Y be a complete intersection of two surfaces S_1 and S_2 with respective degrees a and b lying in $\mathbb{P}^3$, then the geometric genus of Y is given as $g = \frac{1}{2}ab(a + b - 4) + 1$, and the degree of Y as $d = ab$.*

Proof: Let H be the hypersurface of degree a. There is an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-a) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_H \to 0.$$

Twisting by t ($0 \to \mathcal{O}_{\mathbb{P}^n}(t - a) \to \mathcal{O}_{\mathbb{P}^n}(t) \to \mathcal{O}_H(t) \to 0$) and computing dimensions we see that

$$\dim\, \mathcal{O}_H(t) = \dim\, \mathcal{O}_{\mathbb{P}^n}(t) - \dim\, \mathcal{O}_{\mathbb{P}^n}(t - a) = \binom{3 + t}{3} - \binom{3 + t - a}{3}.$$

Furthermore we have the following exact sequence

$$0 \to \mathcal{O}_H(-b) \to \mathcal{O}_H \to \mathcal{O}_Y \to 0.$$

Twisting by $a + b - 4$ yields the exact sequence

$$0 \to \mathcal{O}_H(a - 4) \to \mathcal{O}_H(a + b - 4) \to \mathcal{O}_Y(a + b - 4) \to 0.$$

Applying what was obtained regarding $\dim\, \mathcal{O}_H(t)$ we get the following

$$g(Y) = \dim\, \mathcal{O}_Y = \dim\, \mathcal{O}_Y(a + b - 4) = \dim\, \mathcal{O}_H(a + b - 4) - \dim\, \mathcal{O}_H(a - 4)$$

$$= \binom{a+b-1}{3} - \binom{b-1}{3} - \binom{a-1}{3} + \binom{-1}{3}$$

After performing the necessary simplifications the latter expression becomes $\frac{1}{2}ab(a+b-4)+1$, which is the arithmetic genus $p_a(Y)$ of Y. Since the arithmetic genus and geometric genus coincide for curves, the geometric genus of Y is found to be $g = \frac{1}{2}ab(a+b-4)+1$, as desired. As Y is given as $S_1 \cap S_2$ where $S_1 = Z(f_1)$ with $deg(f_1) = a$, and $S_2 = Z(f_2)$ with $deg(f_2) = b$ we have that the intersection multiplicity of a general plane with Y is the product of its intersection multiplicity with S_1 and S_2, hence finding $deg(Y) = ab$.

To proceed in a different manner, Let S be the homogeneous polynomial ring in four indeterminates x_0, x_1, x_2, x_4 as a graded ring where grading is by degree. Let $S_1 = Z(f_1) \quad I(S_1) = (f_1)$, $S_2 = Z(f_2) \quad I(S_2) = (f_2)$, and $Y = S_1 \cap S_2$ with $\dim(Y) = 1$, and so $I(Y) = (f_1) + (f_2)$. Then we have the exact sequence

$$0 \to S/(f_1)(z-b) \xrightarrow{f_2} S/(f_1) \to S/(f_1)+(f_2) \to 0.$$

Then computing the Hilbert functions resulting from this exact sequence we write

$$\varphi_{S/(f_1)+(f_2)}(z) = \varphi_{S/(f_1)}(z) - \varphi_{S/(f_1)}(z-b).$$

which implies for the Hilbert polynomials that

$$P_Y(z) = P_{S(Y)}(z) = P_{S_1}(z) - P_{S_1}(z-b) =$$

$$\binom{z+n}{n} - \binom{z-a+n}{n} - \binom{z-b+n}{n} + \binom{z-b-a+n}{n} \quad (\dagger).$$

And using the identity $\binom{z-d+n}{n} = (-1)^n \binom{d-z-1}{n}$ we get

$$P_Y(z) = \binom{z+n}{n} + (-1)^{n+1}\binom{a-z-1}{n} + (-1)^{n+1}\binom{b-z-1}{n} + (-1)^n\binom{a+b-z-1}{n}.$$

$$(for\; n = 3\; in\; \mathbb{P}^3) \Rightarrow P_Y(z) = \binom{z+3}{3} + \binom{a-z-1}{3} + \binom{b-z-1}{3} - \binom{a+b-z-1}{3}.$$

$$\Rightarrow P_Y(0) = 1 + \binom{a-1}{3} + \binom{b-1}{3} - \binom{a+b-1}{3}.$$

And by $p_a(Y) = (-1)^{dim(Y)}(1 - P_Y(0)) = \binom{a+b-1}{3} - \binom{a-1}{3} - \binom{b-1}{3}$

$= \frac{1}{2}ab(a+b-4)+1$ which is the arithmetic, and hence geometric genus of Y.

To find the degree of Y in a formal manner, we need to compute the leading coefficient of the Hilbert polynomial $P_Y(z)$ in (†). Now for $n = 3$ we will obtain

$$P_Y(z) = \binom{z+3}{3} - \binom{z-a+3}{3} - \binom{z-b+3}{3} + \binom{z-b-a+3}{3}$$

$$= \frac{(z+3)(z+2)(z+1)}{3!} - \frac{(z-a+3)(z-a+2)(z-a+1)}{3!}$$

$$-\frac{(z-b+3)(z-b+2)(z-b+1)}{3!} + \frac{(z-(a+b)+3)(z-(a+b)+2)(z-(a+b)+1)}{3!}$$

And since $(z+3)(z+2)(z+1) = z^3 + 6z^2 + 11z + 6$, comparing the leading coefficients in the four summands above we find out that the leading coefficient of the polynomial $P_Y(z)$ will be $\frac{ab}{3!}$ and hence the degree of Y will be ab. $\square$

2.4 Hyperelliptic Case

As a rather trivial case, we may note that of the existence of hyperelliptic curves in $\mathbb{P}^3$ with any arbitrary genus $g \in \mathbb{Z}_{\geq 2}$. Let $F(x) \in k[x]$ be a polynomial with no multiple roots and of odd degree $n = 2g+1$. Let $Y \subseteq \mathbb{A}^2$ be the affine plane curve with equation $y^2 = F(x)$. In order to avoid cumbersome situtations assume that char $k \neq 2$. Since the polynomial $y^2 - F(x)$ has degree 2 in y, it is clear that the polynomial $y^2 - F(x)$ is irreducible and hence the affine plane curve with equation $y^2 = F(x)$ is irreducible, then by Noether's normalization theorem (cf. [29] 123-130) there exists a nonsingular projective model X of Y, which is a normal variety sitting in $\mathbb{P}^2$. We will show that such a curve X is a hyperelliptic curve with genus g, computing the canonical class and the genus of X.

We first present a definition which will be used in our computation.

Definition 2.4.1 *Let X and Y be irreducible varieties of the same dimension and $f : X \to Y$ a regular map with the property that $f(X) \subseteq Y$ is dense. The degree of the field extension $f^*(k(Y)) \subseteq k(X)$ is called the degree of f:*

$$deg\ f = [k(X) : f^*(k(Y))].$$

It is a well-known fact that for a finite map $f : X \to Y$ of irreducible varieties with Y a normal variety, the number of inverse images of any point $y \in Y$ is $\leq deg\ f$.

It is easy to check that the affine plane curve Y is nonsingular. If $\overline{Y}$ is the projective closure of Y then X is the normalization of $\overline{Y}$. Obviously the rational map $Y \to \mathbb{A}^1$ defined by $(x, y) \mapsto x$ induces a regular map $f : X \to \mathbb{P}^1$. Clearly $deg\ f = [k(X) : k(\mathbb{P}^1)] = [\frac{k(x,y)}{(y^2 = F(x))} : k(x)] = 2$, so that if $\alpha \in \mathbb{P}^1$ and u is a local parameter at α, the inverse image $f^{-1}(\alpha)$ either consists of two points z', z'' with $v_{z'}(u) = v_{z''}(u) = 1$, or $f^{-1}(\alpha) = z$ with $v_z(u) = 2$. We have a map $\varphi : X \to \overline{Y}$ which is an isomorphism of $\varphi^{-1}(Y)$ and Y. Then it follows that if $\xi \in \mathbb{A}^1$ has coordinate α then

$$f^{-1}(\alpha) = \begin{cases} \{z', z''\} & \text{if } F(\alpha) \neq 0; \\ z & \text{if } F(\alpha) = 0. \end{cases}$$

Now consider the point at infinity $a_\infty \in \mathbb{P}^1$. If x denotes the coordinate on $\mathbb{P}^1$ then a local parameter at a_∞ is $u = X^{-1}$. If $f^{-1}(a_\infty) = \{z', z''\}$ consisted of 2 points then u would be a local parameter at the point z', say. Then it would follow that $v_{z'}(u) = 1$ and hence $v_{z'}(F(x)) = -n$; but as $y^2 = F(x)$, we have $v_{z'}(F(x)) = 2v_{z'}(y)$, and this contradicts the hypothesis that n is odd. Thus $f^{-1}(a_\infty)$ consists of just one point z_∞, and $v_{z_\infty}(x) = -2$, $v_{z_\infty}(y) = -n$. Then it follows that $X = \varphi^{-1}(Y) \cup z_\infty$.

We proceed to examine the differential forms on X. Consider, for instance, the form $\omega = \frac{dx}{y}$. At a point $\xi \in Y$, if $y(\xi) \neq 0$ then x is a local parameter, and $v_\xi(\omega) = 0$. If $y(\xi) = 0$ then y in this case is a local parameter, and

$v_\xi(x) = 2$, so that it again follows that $v_\xi(\omega) = v_\xi(dx) - v_\xi(y) = 1 - 1 = 0$. thus $div\ \omega = kz_\infty$, and it remains to determine the value of k. For this, it is enough to recall that if t is a local parameter at z_∞ then $x = t^{-2}u$ and $y = t^{-n}v$, where $u, v, u^{-1}, v^{-1} \in \mathcal{O}_{z_\infty}$, and therefore $div\ \omega = (n-3)z_\infty = (2g-2)z_\infty$.

Now we attempt to determine $\Omega^1[X]$. On Y as $2ydy - \frac{\partial F}{\partial x}dx = 0 \Rightarrow \frac{dx}{y} = \frac{2dy}{F'(x)}$ holds as an identity ω is a basis of the module $\Omega^1[Y]$, that is, $\Omega^1[Y] = k[Y]\omega$, so that any form in $\Omega^1[X]$ is of the form $u\omega$, where $u \in k[Y]$, and hence u is of the form $P(x) + Q(x)y$ with $P, Q \in k[X]$. It remains to check when these forms are regular at z_∞. This happens if and only if

$$v_{z_\infty}(u) \geq -(n-3).$$

We find all such $u \in k[Y]$. Since $v_{z_\infty}(x) = -2$, it follows that $v_{z_\infty}(P(x))$ is always even and since $v_{z_\infty}(y) = -n$, that $v_{z_\infty}(Q(x)y)$ is always odd. Hence

$$v_{z_\infty}(u) = v_{z_\infty}(P(x) + Q(x)y) \leq \min\{v_{z_\infty}(P(x)), v_{z_\infty}(Q(x)y)\}$$

and so if $Q \neq 0$ we have $v_{z_\infty}(u) \leq -n$. Hence $u = P(x)$ and $v_{z_\infty}(u) \geq -(n-3)$ gives $2\deg P \leq n-3$, that is, $\deg P \leq g-1$, whence $n = 2g+1$.

We have found out that $\Omega^1[X]$ consists of forms $P(x)dx/y$ where the degree of $P(x)$ is $\leq g-1$. It follows that the genus of X is $g(X) = h^1(X) = \dim\Omega^1[X] = g$.

So there exists a hyperelliptic curve of any arbitrary genus $g \in \mathbb{Z}_{\geq 2}$. And since a plane curve of degree n has genus $\binom{n-1}{2}$ we also observe that not every hyperelliptic curve is a plane curve, e.g. a hyperelliptic curve as in the above mentioned construction with genus 2 and degree 5 is not a plane curve as $\binom{4}{2} = 6 \neq 2$. Indeed a hyperelliptic curve of genus $g \in \mathbb{Z}_{\geq 2}$ and degree $2g+1$ is a plane curve implies that $g = \binom{2g+1}{2} = g(2g-1) \Leftrightarrow g = 1 \notin \mathbb{Z}_{\geq 2}$, a contradiction. Hence any hyperelliptic curve appearing in the above construction is not a plane curve, and hence does not lie on any linear subspace of $\mathbb{P}^3$.

Remark: Observe that if X is a curve of genus $g = 2$, then the canonical divisor defines a complete linear system $|K|$ of degree 2 and dimension 1, without base points. It is clear that $deg(K) = 2g - 2 = 2$, and $dim(|K|) = l(K) - 1 = 1$. Suppose P is a base point of $|K|$, then $l(K - P) = l(K) = 2$ by definition. Then by Riemann-Roch, $l(P) = 2 + 2 - 2 = 2$. Thus there exist a non-constant rational function f with a pole at P of order 1 and regular everywhere else. Then naturally f defines an isomorphism from X to $\mathbb{P}^1$, resulting in a contradiction since X has genus 2 not 0. Therefore $|K|$ has no base points. Then (cf. [13] Remark 7.8.1, pg. 158) there is a finite morphism $f : X \to \mathbb{P}^1$ with degree equal to $deg(K) = 2$. Therefore X must be a hyperelliptic curve.

Remark: There are also some other facts which are worth mentioning. For any $d \geq \frac{1}{2}g + 1$, any curve of genus g has a g^1_d (a base point linear system of divisors with degree d and dimension 1); for $d < \frac{1}{2}g + 1$, there exist curves of genus g having no g^1_d. In particular, this implies that there exist nonhyperelliptic curves of every genus $g \geq 3$. Indeed, for any $n > e \geq 0$, let X be the rational scroll of degree $d = 2n - e$ in $\mathbb{P}^{d+1}$ which is an embedding of the rational ruled surface X_e in such a way that all the fibres f have degree 1. If $n \geq 2e - 2$, X contains a nonsingular curve Y of genus $g = d + 2$ which is a canonical curve in this embedding. Moreover, for every $g \geq 4$ there exists a nonhyperelliptic curve of genus g which has a g^1_3. (For the proofs and discussion, cf. [19] and [13] Remark 5.5.1, page 345)

2.5 Curves on Surfaces of Low Degree in $\mathbb{P}^3$

2.5.1 Curves on a Linear Subspace of $\mathbb{P}^3$

It is quite natural to start considering curves lying on surfaces, and estimating the degree of such an ambient surface for suitable degree-genus pairs (d, g) culminates a new direction in classification. Consideration of linear systems of divisors with certain degrees and dimensions on such surfaces creates a new road in mathematical interest. For surfaces of low degree; namely degree 1, 2, and 3 the problem has a simple answer.

A surface S of degree 1 is a linear subspace of $\mathbb{P}^3$, and hence isomorphic to $\mathbb{P}^2$. If a curve $\mathcal{C}$ of degree d lies on such a surface, it can be mapped isomorphically to $\mathbb{P}^2$, then by the famous degree-genus formula its genus g is given by

$$g = \binom{d-1}{2} = \frac{1}{2}(d-1)(d-2)$$

which can be verified by the adjunction formula, if a curve $\mathcal{C}$ of degree d lies in $X = \mathbb{P}^2$ then $2g-2 = d(d-3) \Rightarrow g = \frac{1}{2}(d-1)(d-2)$, or which can be calculated independently as follows:

Let $\mathcal{C} \subseteq \mathbb{P}^2_k$ be a curve of degree d. Then $\mathcal{C}$ is a closed subscheme defined by a single homogeneous polynomial $f(x_0, x_1, x_2)$ of degree d, thus

$$\mathcal{C} = Proj(S/(f)).$$

Now we explicitly compute $p_a(\mathcal{C})$. Let $I = (f)$ with deg $f = d$. Then

$$1 - p_a = h_0(\mathcal{O}_\mathcal{C}) - h_1(\mathcal{O}_\mathcal{C}) + h_2(\mathcal{O}_\mathcal{C}) = \chi(\mathcal{O}_\mathcal{C}).$$

We have an exact sequence

$$0 \to \mathcal{I}_\mathcal{C} \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_\mathcal{C} \to 0.$$

Now $\mathcal{I}_\mathcal{C} \cong \mathcal{O}_{\mathbb{P}^2}(-d)$ as $\mathcal{O}_{\mathbb{P}^2}(-d)$ can be considered to be generated by $\frac{1}{f}$ on $D_+(f)$ and by something else elsewhere, and then multiplication by f gives an inclusion $\mathcal{O}_{\mathbb{P}^2}(-d) \mid_{D_+(f)} \hookrightarrow \mathcal{O}_{\mathbb{P}^2} \mid_{D_+(f)}$, etc. Therefore

$$\chi(\mathcal{O}_\mathcal{C}) = \chi(\mathcal{O}_{\mathbb{P}^2}) - \chi(\mathcal{O}_{\mathbb{P}^2}(-d)).$$

Now

$$\chi(\mathcal{O}_{\mathbb{P}^2}) = h^0(\mathcal{O}_{\mathbb{P}^2}) - h^1(\mathcal{O}_{\mathbb{P}^2}) + h^2(\mathcal{O}_{\mathbb{P}^2}) = 1 + 0 + 0 = 1$$

and

$$\chi(\mathcal{O}_{\mathbb{P}^2}(-d)) = h^0(\mathcal{O}_{\mathbb{P}^2}(-d)) - h^1(\mathcal{O}_{\mathbb{P}^2}(-d)) + h^2(\mathcal{O}_{\mathbb{P}^2}(-d)) = 0 + 0 + \frac{1}{2}(d-1)(d-2).$$

For the last computation we used duality to see that

$$h^2(\mathcal{O}_{\mathbb{P}^2}(-d)) = h^0(\mathcal{O}_{\mathbb{P}^2}(d-3)) = \dim\, S_{d-3} = \frac{1}{2}(d-1)(d-2).$$

Thus $\chi(\mathcal{O}_{\mathcal{C}}) = 1 - \frac{1}{2}(d-1)(d-2)$ and so

$$g = p_a(\mathcal{C}) = \frac{1}{2}(d-1)(d-2).$$

Therefore there is not much freedom for the degree-genus pairs (d, g) for curves lying on a surface of degree 1.

2.5.2 Curves on a Nonsingular Quadric Surface

Let $\mathbb{P}^N = \mathbb{P}^{(n+1)(m+1)-1}$ be projective space with homogeneous coordinates $z_{i,j}$, $0 \leqq i \leqq n, 0 \leqq j \leqq m$. There is an obviously well-defined set-theoretic map $f : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ given by $f(x_i, y_j) = z_{i,j}$, which is called the *Segre embedding* (cf. [7] pp. 43-44).

Lemma 2.5.1 *: Let $f : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ be the above-defined map. Then:*

(i) The image $X = f(\mathbb{P}^n \times \mathbb{P}^m)$ is a projective variety in $\mathbb{P}^N$, with ideal generated by the homogeneous polynomials $z_{i,j}z_{i',j'} - z_{i,j'}z_{i',j}$ for all $0 \leq i, i' \leq n$ and $0 \leq j, j' \leq m$.

(ii) The map $f : \mathbb{P}^n \times \mathbb{P}^m \to X$ is an isomorphism. In particular, $\mathbb{P}^n \times \mathbb{P}^m$ is a projective variety.

(iii) The closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$ are exactly those subsets that can be written as the zero locus of polynomials in $k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ that are bihomogeneous in the x_i and y_j.

Proof:
(i): It is obvious that the points of $f(\mathbb{P}^n \times \mathbb{P}^m)$ satisfy the given equations. Conversely, let P be a point in $\mathbb{P}^n$ with coordinates $z_{i,j}$ that satisfy the given equations. At least one of these coordinates must be non-zero; we can assume without loss of generality that it is $z_{0,0}$. Let us pass to the affine coordinates by setting $z_{0,0} = 1$. Then we have $z_{i,j} = z_{i,0}z_{0,j}$; so by setting $x_i = z_{i,0}$ and $y_j = z_{0,j}$ we obtain a point of $\mathbb{P}^n \times \mathbb{P}^m$ that is mapped to P by f.

(ii): Continuing the above notation, let $P \in f(\mathbb{P}^n \times \mathbb{P}^m)$ be a point with $z_{0,0} = 1$. If $f(x_i, y_j) = P$, it follows that $x_0 \neq 0$ and $y_0 \neq 0$, so we can assume $x_0 = 1$ and $y_0 = 1$ as the x_i and y_j are only determined up to a common scalar. But then it follows that $x_i = z_{i,0}$ and $y_j = z_{0,j}$; i.e. f is bijective.

The same calculation shows that f and f^{-1} are given (locally in affine coordinates) by polynomial maps; so f is an isomorphism.

(iii): It follows by the isomophism of (ii) that any closed subset of $\mathbb{P}^n \times \mathbb{P}^m$ is the zero locus of homogeneous polynomials in the $z_{i,j}$, i.e. of bihomogeneous polynomials in the x_i and y_j (of the same degree). Conversely, a zero locus of bihomogeneous polynomials can always be written as a zero locus of bihomogeneous polynomials of the same degree in the x_i and y_j (since $Z(f) = Z({x_0}^d f, \ldots, {x_n}^d f)$ for all homogeneous polynomials f and every $d \geq 0$). But such a polynomial is obviously a polynomial in the $z_{i,j}$, so it determines an algebraic set in $X \cong \mathbb{P}^n \times \mathbb{P}^m$.
□

Let $Q \subseteq \mathbb{P}^3$ be a quadric surface. Then the variety Q has dimension 2 in $\mathbb{P}^3$. Now let us pass by dehomogenization if necessary to suitable affine spaces $\mathbb{A}^3 \subseteq \mathbb{P}^3$. Considering the affine variety Q in $\mathbb{A}^3$, we see that Q must be the zero locus of a single polynomial. The reasoning is as follows: Let $I(Q) = \mathfrak{I}$, Since $\dim\, \mathfrak{I} + \dim\, k[x_0, x_1, x_2]/\mathfrak{I} = \dim\, k[x_0, x_1, x_3] = 3 \Leftrightarrow \dim\, \mathfrak{I} + \dim\, Q = 3 \Leftrightarrow \dim\, \mathfrak{I} = 3 - 2 = 1$. Then by Krull's Hauplidealsatz for any $f \in \mathfrak{I}$ which is neither a zero divisor nor a unit, every prime ideal $\mathfrak{p}$ containing f has height

1. Moreover in a Noetherian integral domain A, as is the case with $k[x_0, x_1, x_2]$, A is a unique factorization domain if and only if every prime ideal of height 1 is principal. Hence any prime ideal $\mathfrak{p}$ containing f has height 1 and for this reason it must be equal to $\mathfrak{I}$ and since $k[x_0, x_1, x_2]$ is a UFD $\mathfrak{I}$ must be a principal ideal, and hence we must have $\mathfrak{I} = (f)_{ideal}$. Now let us go back to $\mathbb{P}^3$, we know that $Q = Z(f)$ for some $f \in S[x_0, x_1, x_2, x_3]$ with $\deg f = 2$. Then letting $X = (x_0\ x_1\ x_2\ x_3)$, f can be written in the form $f = XAX^t$ for a suitable matrix $A \in \mathbb{O}(4, k)$, i.e. $A = A^t$. Then by linear algebra the symmetric matrix A is diagonalizable, i.e. there is a matrix $P \in \mathbb{GL}(4, k)$ such that $P^{-1}AP = \Lambda$ where Λ is a diagonal matrix, and $P^t = P^{-1}$. Then since the function $\varphi : \mathbb{P}^3 \to \mathbb{P}^3$ defined by $(x_0\ x_1\ x_2\ x_3) \mapsto (x_0\ x_1\ x_2\ x_3)P^{-1} = (y_0\ y_1\ y_2 y_3) = Y$ is clearly an isomorphism, we get that $f(Y) = YAY^t = XP^{-1}APX^t = X\Lambda X^t$, a perfect square. Moreover if we impose the extra condition of nonsingularity we observe that all nonsingular quadric surfaces are isomorphic. For this reason we as a model take the equation of our nonsingular quadric surface to be $xy - zw = 0$ in $\mathbb{P}^3$.

By Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, a quadric surface Q satisfies $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. $\mathbb{P}^1 \times \mathbb{P}^1$ has two families of lines on it, namely $P \times \mathbb{P}^1$ and $\mathbb{P}^1 \times P$ for any chosen $P \in \mathbb{P}^1$. Let p_1, p_2 be the projections $p_1 : Q \to \varphi^{-1}(\mathbb{P}^1 \times P)$, $p_2 : Q \to \varphi^{-1}(P \times \mathbb{P}^1)$and where $\varphi : Q \to \mathbb{P}^1 \times \mathbb{P}^1$ is the isomorphism of Q onto $\mathbb{P}^1 \times \mathbb{P}^1$, and P is the any chosen point, e.g. take $P = [0 : 1]$. Then clearly p_i are seen to be morphisms, since they are simply projections, hence polynomials maps (followed by an isomorphism). Hence for any regular map $f : \mathbb{P}^1 \times \mathbb{P}^1 \to k$, the pull-back of f under p_i, i.e. $p_i^* f = f \circ p_i : Q \to k$ are regular maps. Then $p_1^*, p_2^* : Cl(\mathbb{P}^1 \times \mathbb{P}^1) \to ClQ$ defined by $\sum n_j Y_j \mapsto \sum n_j p_i^{-1}(Y_j)$ are naturally homomorphisms of the divisor class groups. Since if $f \in Div(\mathbb{P}^1 \times \mathbb{P}^1)$ then $p_i^*((f)) = f \circ p_i$ is the divisor of $\varphi^* f$ which is viewed as an element of $K(Q)$. First we show that p_1, p_2 are injective. Let $Y = P \times \mathbb{P}^1$. Then $Q \backslash P = \mathbb{A}^1 \times \mathbb{P}^1$, and the composition

$$Cl(\mathbb{P}^1) \xrightarrow{p_2} Cl(Q) \to Cl(\mathbb{A}^1 \times \mathbb{P}^1)$$

is an isomorphism. Hence p_1^*, (and hence p_2^*) are injective. for any prime divisor Y on Q, $Y \cap U$ is either empty or a prime divisor on U, for any open subset with property $U = Q\backslash Z$, where Z is a proper closed subset of Q . If f is a divisor on Q, and $(f) = \sum n_j Y_j$, then by viewing f as a rational function on U, we have $(f)_U = \sum n_j (Y_j \cap U)$, so indeed we have a homomorphism $Cl(Q) \to Cl(U)$, for any open subset $U = Q\backslash Z$. The kernel of $Cl(Q) \to Cl(U)$ consists of divisors whose supports are directly contained in Z. If Z is irreducible, the kernel is just the subgroup of $Cl(Q)$ which is generated by $1 \cdot Z$. Hence applying with $Z = Y$ in our case, by the help of the exact sequence

$$\mathbb{Z} \to Cl(Q) \to Cl(\mathbb{A}^1 \times \mathbb{P}^1) \to 0.$$

yields that in this sequence the first map sends 1 to Y. But $Cl(\mathbb{P}^1)$ is usually identified with $\mathbb{Z}$ by letting 1 be the class of a point, then first map is just p_1^*, hence is injective. Since the image of p_2^* goes isomorphically to $Cl(\mathbb{A}^1 \times \mathbb{P}^1)$ as we have just observed, we reach the conclusion that $Cl(Q) \cong Im(p_1^*) \oplus Im(p_2^*) = \mathbb{Z} \oplus \mathbb{Z}$. If D is any divisor on Q, let (a, b) be the ordered pair of integers in $\mathbb{Z} \oplus \mathbb{Z}$ corresponding to the class of D under this isomorphism. Then we say that D is type (a, b) on Q.

Thus Q has two families of lines on it, and the intersection number with lines of these two families produce two numbers, say a, and b, hence a bidegree (a, b) for a curve C. Picard group of Q, $Pic(Q)$, which is the equivalence class of divisors defined on Q, is a free group on two generators; so $Pic(Q) \cong \mathbb{Z} \oplus \mathbb{Z}$. By this correspondence, the class of such a curve C is represented by a pair of integers (a, b) by which the degree and genus of the curve C can be calculated as

$$d = a + b,$$

$$g = (a - 1)(b - 1).$$

Let Q be the nonsingular quadric surface in $\mathbb{P}^3_k$ with equation $xy = zw$ over a field k. Consider locally principal closed subschemes Y of Q. These subshemes will correspond to Cartier divisors on Q. Since we know that Pic$Q \cong Z \oplus Z$, we simply may talk about the type (a, b) of Y. Let us denote the invertible sheaf $\mathcal{L}(Y)$ by $\mathcal{O}_Q(a, b)$. Hence for any $n \in \mathbb{Z}$, $\mathcal{O}_Q(n) = \mathcal{O}_Q(n, n)$. We will use the special case $(q, 0)$ and $(0, q)$, with $q > 0$, when Y is a disjoint union of q lines $\mathbb{P}^1$ in Q to show that (cf. [31] pp. 86-91):

1. if $\mid a-b \mid \leq 1$, then $H^{1}(Q, \mathcal{O}_Q(a,b)) = 0$;
2. if $a, b < 0$, then $H^{1}(Q, \mathcal{O}_Q(a,b)) = 0$;
3. if $a \leq -2$, then $H^{1}(Q, \mathcal{O}_Q(a,0)) \neq 0$.

Before proving the above stated results, we will need a rather big lemma in which explicit computation of $H^{1}(Q, \mathcal{O}_Q(0,-q))$ and some other technical staff is carried out.

Lemma 2.5.2 *Let $q > 0$, then*

$$dim_k(H^{1}(Q, \mathcal{O}_Q(-q,0)) = H^{1}(Q, \mathcal{O}_Q(0,-q)) = q-1.$$

Furthermore, all terms in the long exact sequence of cohomology associated with the short exact sequence

$$0 \to \mathcal{O}_Q(-q,0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$$

are known.

Proof: We prove the lemma for $\mathcal{O}_Q(-q,0)$ only, since the argument for $\mathcal{O}_Q(0,-q)$ is exactly a direct replica. Suppose Y is the disjoint union of q lines $\mathbb{P}^1$ in Q so $\mathcal{I}_Y = \mathcal{O}_Q(-q,0)$. The sequence

$$0 \to \mathcal{O}_Q(-q,0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$$

is exact. The associated long exact sequence of cohomology to this short exact sequence is

$$0 \to \Gamma(Q, \mathcal{O}_Q(-q,0)) \to \Gamma(Q, \mathcal{O}_Q) \to \Gamma(Q, \mathcal{O}_Y)$$
$$\to H^{1}(Q, \mathcal{O}_Q(-q,0)) \to H^{1}(Q, \mathcal{O}_Q) \to H^{1}(Q, \mathcal{O}_Y)$$
$$\to H^{2}(Q, \mathcal{O}_Q(-q,0)) \to H^{2}(Q, \mathcal{O}_Q) \to H^{2}(Q, \mathcal{O}_Y) \to 0$$

We can compute all of the terms in the above long exact sequence. But for the purpose at hand it is sufficient to view the summands as k-vector spaces so we systematically do this throughout. Since $\mathcal{O}_Q(-q,0) = \mathcal{I}_Y$ is the ideal sheaf of Y, its global sections must vanish on Y. But $(I)_Y$ is a subsheaf of $\mathcal{O}_Q$ whose global sections are only constants. Since the only constant which vanishes on Y is 0, $\Gamma(Q, \mathcal{O}_Q(-q, 0)) = 0$. And $\Gamma(Q, \mathcal{O}_Y) = k$. Since Y is the disjoint union of q copies of $\mathbb{P}^1$ and each copy has global sections k, $\Gamma(Q, \mathcal{O}_Y) = k^{\oplus q}$. Since Q is a complete intersecion of dimension 2, $H^1(Q, \mathcal{O}_Q) = 0$. Because Y is isomorphic to several copies of $\mathbb{P}^1$, the general result that $H_*^n(\mathcal{O}_{\mathbb{P}^n}) = \{\sum a_I X_I$: entries in I negative $\}$ implies that $H^1(Q, \mathcal{O}_Y) = H^1(Y, \mathcal{O}_Y) = 0$. Since Q is a hypersurface of degree 2 in $\mathbb{P}^3$, $p_a(Q) = 0$. Thus $H^2(Q, \mathcal{O}_Q) = 0$. Combining the stated facts and some basic properties of exact sequences show that $H^1(Q, \mathcal{O}_Q(-q, 0)) = k^{\oplus(q-1)}$, $H^2(Q, \mathcal{O}_Q(-q, 0)) = 0$ and $H^2(Q, \mathcal{O}_Y) = 0$. Our long exact sequence is now

$$0 \to \Gamma(Q, \mathcal{O}_Q(-q, 0)) = 0 \to \Gamma(Q, \mathcal{O}_Q) = k \to \Gamma(Q, \mathcal{O}_Y) = k^{\oplus q}$$

$$\to H^1(Q, \mathcal{O}_Q(-q, 0)) = k^{\oplus(q-1)} \to H^1(Q, \mathcal{O}_Q) = 0 \to H^1(Q, \mathcal{O}_Y) = 0$$

$$\to H^2(Q, \mathcal{O}_Q(-q, 0)) = 0 \to H^2(Q, \mathcal{O}_Q) = 0 \to H^2(Q, \mathcal{O}_Y) = 0 \to 0$$

□.

Now the statement numbered by (3) follows immediately from the lemma because

$$H^1(Q, \mathcal{O}_Q(a, 0)) = k^{\oplus(-a-1)} \neq 0$$

for $a \leq -2$.

We compute now (1) and (2) explicitly. Let a be an arbitrary integer. First we show that $\mathcal{O}_Q(a, a) = 0$. We have a short exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-2) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_Q \to 0$$

where the first map is the multiplication by $xy - zw$. Twisting by a gives an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^3}(-2 + a) \to \mathcal{O}_{\mathbb{P}^3}(a) \to \mathcal{O}_Q(a) \to 0.$$

The long exact sequence of cohomology yields an exact sequence

$$\ldots \to H^1(\mathcal{O}_{\mathbb{P}^3}(a)) \to H^1(\mathcal{O}_Q(a)) \to H^2(\mathcal{O}_{\mathbb{P}^3}(-2 + a)) \to \ldots$$

But from the explicit computations of projective space it follows that $H^1(\mathcal{O}_{\mathbb{P}^3}(a)) = 0$ and $H^2(\mathcal{O}_{\mathbb{P}^3}(-2+a)) = 0$ whence leading to the conclusion that $H^1(\mathcal{O}_Q(a)) = 0$.
Next we show that $\mathcal{O}_Q(a-1,a) = 0$. Let Y be a single copy of $\mathbb{P}^1$ sitting in Q so that Y has type $(1,0)$. Then we have an exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0.$$

But $\mathcal{I}_Y = \mathcal{O}_Q(-1,0)$ so this exact sequence transforms into the form

$$0 \to \mathcal{O}_Q(-1,0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0.$$

Twisting by a at this stage yields the exact sequence

$$0 \to \mathcal{O}_Q(a-1,a) \to \mathcal{O}_Q(a) \to \mathcal{O}_Y(a) \to 0.$$

The long exact sequence of cohomology associated to this sequence gives an exact sequence

$$\ldots \to \Gamma(\mathcal{O}_Q(a-1,a)) \to \Gamma(\mathcal{O}_Y(a)) \to H^1(\mathcal{O}_Q(a-1,a)) \to H^1(\mathcal{O}_Q(a)) \to \ldots$$

We just showed that $H^1(\mathcal{O}_Q(a)) = 0$, so to see that $H^1(\mathcal{O}_Q(a-1,a)) = 0$ it is sufficient to note that the map $\Gamma(\mathcal{O}_Q(a)) \to \Gamma(\mathcal{O}_Y(a))$ is surjective. This can be visualized by simply noting that $Q = Proj(k[x,y,z,w]/(xy-zw))$ and without loss of generality $Y = Proj(k[x,y,z,w]/(xy-zw,x,z))$ and observing that the degree a part of $k[x,y,z,w]/(xy-zw)$ surjects onto the degree a part of $k[x,y,z,w]/(xy-zw,x,z)$. Thus $H^1(\mathcal{O}_Q(a,a-1)) = 0$. This yields the statement (1).

As for the statement (2), it suffices to show that for $a > 0$,

$$H^1(\mathcal{O}_Q(-a,-a-n)) = H^1(\mathcal{O}_Q(-a-n,-a)) = 0$$

for all $n > 0$. Thus let $n > 0$ and suppose Y is a disjoint union of n copies of $\mathbb{P}^1$ in such a way that $\mathcal{I}_Y = \mathcal{O}_Q(0,-n)$. Then we have an exact sequence

$$0 \to \mathcal{O}_Q(0,-n) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0.$$

Twisting by $-a$ yields the exact sequence

$$0 \to \mathcal{O}_Q(-a, -a-n) \to \mathcal{O}_Q(-a) \to \mathcal{O}_Y(-a) \to 0.$$

The long exact sequence of cohomology associated to the last obtained short exact sequence gives us the following exact sequence

$$\ldots \to \Gamma(\mathcal{O}_Y(-a)) \to H^1(\mathcal{O}_Q(-a, -a-n)) \to H^1(\mathcal{O}_Q(-a) \to \ldots$$

As a commonly known fact, since Y is just several copies of $\mathbb{P}^1$ and $-a < 0$, $\Gamma(\mathcal{O}_Y(-a)) = 0$. Since we also computed above, $H^1(\mathcal{O}_Q(-a)) = 0$. Thus $H^1(\mathcal{O}_Q(-a, -a-n)) = 0$, as required. Showing that $H^1(\mathcal{O}_Q(-a-n, -a)) = 0$ is exactly the same.

And using this result we will prove that:

Proposition 2.5.1

1. *If Y is a locally principal closed subscheme of type (a, b) with $a, b > 0$, then Y is connected.*
2. *Assuming k to be algebraically closed, for any $a, b > 0$, there exists an irreducible nonsingular curve Y of type (a, b).*
3. *An irreducible nonsingular curve Y of type (a, b), $a, b > 0$ on Q is projectively normal (its homogeneous coordinate ring $S(Y)$ is integrally closed) if and only if $\mid a - b \mid \leq 1$. In particular, this gives lots of examples of nonsingular, but not projectively normal curves in $\mathbb{P}^3$, corresponding to locally principal closed subschemes of type (a, b) where $|a - b| > 1$. The simplest is the one of type $(1, 3)$ which is just the rational quartic curve*

Proof: **1.** Computing the long exact sequence associated to the short exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$$

gives the exact sequence

$$0 \to \Gamma(Q, \mathcal{I}_Y) \to \Gamma(Q, \mathcal{O}_Q) \to \Gamma(Q, \mathcal{O}_Y) \to H^1(Q, \mathcal{I}_Y) \to \ldots$$

But,$\Gamma(\mathcal{I}_Y) = 0$, $\Gamma(Q, \mathcal{O}_Q) = k$, and by (a)-2. above $H^1(Q, \mathcal{I}_Y) = H^1(Q, \mathcal{O}_Q(-a,-b)) = 0$. Thus we have an exact sequence

$$0 \to 0 \to k \to \Gamma(\mathcal{O}_Y) \to 0 \to \dots$$

from which we conclude that $\Gamma(\mathcal{O}_Y) = k$ whence Y is concluded to be connected.
2. Given $(a,b) > 0$, since $Pic(Q) \cong \mathbb{Z} \oplus \mathbb{Z}$, Consider an a-uple embedding $\mathbb{P}^1 \to \mathbb{P}^a$ and a b-uple embedding $\mathbb{P}^1 \to \mathbb{P}^b$. Taking their product, and following with a Segre embedding, we obtain a closed immersion

$$Q \cong \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^a \times \mathbb{P}^b \to \mathbb{P}^n,$$

corresponding to an invertible sheaf $\mathcal{O}_Q(-a,-b)$ of type (a,b) on Q. By Bertini's theorem there is a hyperplane H in $\mathbb{P}^n$ such that the hyperplane section of the (a,b) embedding of Q in $\mathbb{P}^n$ is nonsingular. Pull this hyperplane section back to a nonsingular curve Y of type (a,b) on Q in $\mathbb{P}^3$. Then by part 1 of the proposition Y is connected. Since Y comes from a hyperplane section, by Bertini's theorem, Y is irreducible.
3. An irreducible curve Y of type (a,b) with $a,b > 0$, on Q is projectively normal if and only if $|a-b| \leq 1$. In particular this gives lots of examples of nonsingular, but not projectively normal curves in $\mathbb{P}^3$. The simplest is the one of type $(1,3)$ which is just the rational quartic curve.

Proof: Let Y be an irreducible nonsingular curve of type (a,b). Then the maps

$$\Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) \to \Gamma(Y, \mathcal{O}_Y(n))$$

are surjective for all $n \geq 0$ if and only if Y is projectively normal. To determine when this occurs we have to replace $\Gamma(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n))$ with $\Gamma(Q, \mathcal{O}_Q(n))$. It is easy to see that the above criterion implies that we can make this replacement if Q is projectively normal. Since $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$ is locally isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1 \cong \mathbb{A}^2$ which is normal, we see that Q is normal. Then since Q i sa complete intersection which is normal, Q is projectively normal.
Consider the exact sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_Q \to \mathcal{O}_Y.$$

Twisting by n gives an exact sequence

$$0 \to \mathcal{I}_Y(n) \to \mathcal{O}_Q(n) \to \mathcal{O}_Y(n).$$

Taking cohomology yields the exact sequence

$$\ldots \to \Gamma(Q, \mathcal{O}_Q(n)) \to \Gamma(Q, \mathcal{O}_Y(n)) \to H^1(Q, \mathcal{I}_Y(n)) \to \ldots$$

Thus Y is projectively normal precisely if $H^1(Q, \mathcal{I}_Y(n)) = 0$ for all $n \geq 0$. When can this happen? We apply our computations carried out in part (a). Since $\mathcal{O}_Q(n) = \mathcal{O}_Q(n, n)$,

$$\mathcal{I}_Y(n) = \mathcal{O}_Q(-a, -b)(n) = \mathcal{O}_Q(-a, -b) \otimes_{\mathcal{O}_Q} \mathcal{O}_Q(n, n) = \mathcal{O}_Q(n - a, n - b)$$

If $|a - b| \leq 1$ then $|(n - a) - (n - b)| \leq 1$ for all n so

$$H^1(Q, \mathcal{O}_Q(-a, -b)(n)) = 0$$

for all n which implies Y is projectively normal. On the other hand, if $|a - b| > 1$ let n be the minimum of a and b, without loss of generality b is the minimum, so $n = b$. then from (a) we see that

$$\mathcal{O}_Q(-a, -b)(n) = \mathcal{O}_Q(-a, -b)(b) = \mathcal{O}_Q(-a + b, 0) \neq 0$$

since $-a + b \leq -2$. □

□

2.5.3 Curves on a Nonsingular Cubic Surface

Let $X \subseteq \mathbb{P}^3$ be a nonsingular cubic surface. Then X contains a line L (cf. [29] Theorem-10 on pp. 80). We will present some basic properties of nonsingular cubic surfaces with references to [13] wherever necessary. Let $P_1, \ldots, P_6$ be six points of the plane, with no three of the points collinear, and not all six lying on a conic. Let $\mathfrak{d}$ be the linear system of plane cubic curves through $P_1, \ldots, P_6$, and let X be the nonsingular cubic curve obtained by an embedding of X', which is obtained from $\mathbb{P}^2$ by blowing up $P_1, \ldots, P_6$ into $\mathbb{P}^3$. Thus X is isomorphic to $\mathbb{P}^2$ with 6 points $P_1, \ldots, P_6$ blown up (For the details of this construction, see [13] pp. 395-401). Hence showing the total transform of a line in $\mathbb{P}^2$ by l, and the exceptional divisors resulting from the blowing up of P_i by e_i we have the following important characterization of X (cf. [13], pg. 401):

Proposition 2.5.2 *Let X be the cubic surface whose construction is depicted above. Then:*

(a) $\text{Pic}X \cong \mathbb{Z}^7$, *generated by* $l, e_1, \ldots, e_6$;

(b) *the intersection pairing on X is given by* $l^2 = 1$, $e_i^2 = -1$, $l \cdot e_i = 0$, $e_i \cdot e_j = 0$ *for* $i \neq j$;

(c) *the hyperplane section h is equal to* $3l - \sum e_i$;

(d) *the canonical class is* $K = -h = -3l + \sum e_i$;

(e) *If D is any effective divisor on X, $D \sim al - \sum b_i e_i$, then the degree of D, as a curve in $\mathbb{P}^3$ is given by*

$$d = 3a - \sum b_i;$$

(f) *the self-intersection of D is* $D^2 = a^2 - \sum b_i^2$;

(g) *the arithmetic genus of D is*

$$p_a(D) = \frac{1}{2}(D^2 - d) + 1 = \frac{1}{2}(a-1)(a-2) - \frac{1}{2}\sum b_i(b_i - 1).$$

Proof: The proof of this proposition is given in [13] page 402. □

Also being of special importance we present the following lemmas from [1]:

Lemma 2.5.3 *Let $D \sim al - \sum b_i e_i$ be a divisor class on the cubic surface $X \subseteq \mathbb{P}^3$, and suppose that $b_1 \geq b_2 \geq \ldots \geq b_6 > 0$ and $a \geq b_1 + b_2 + \ldots + b_5$. Then the divisor D is very ample.*

Proof: For the proof, see [13] page 405. □

Recall that a divisor D is called very ample in case the corresponding linear space $\mathcal{L}(D)$ is isomorphic to $\mathcal{O}_X(1)$ for some immersion of X in a projective space, and D is called ample if for every coherent sheaf $\mathfrak{F}$ on X, the sheaf $\mathfrak{F} \otimes \mathcal{L}(D)^n$ is generated by global sections for $n >> 0$. Note that although being ample is an intrinsic property, being very ample depends upon the projective embedding of D. Surprisingly, for divisors on a nonsingular cubic surface in $\mathbb{P}^3$ the concepts of being ample and being very ample coincide with each other (cf. [13] page 405). The following corollary of the above presented lemma is the one related to our classification effort:

Corollary 2.5.1 *Let $D \sim al - \sum b_i e_i$ be a divisor on the nonsingular cubic surface X. Then:*

(a) D is ample $\Leftrightarrow$ very ample $\Leftrightarrow$ $b_i > 0$ for each i, and $a > b_i + b_j$ for each i, j, and $2a > \sum_{i \neq j} b_i$ for each j;

(b) in any divisor class satisfying the conditions of part (a), there is an irreducible nonsingular curve.

Proof: See [13] page 406. □

Example: (cf. [13] pg. 406) Hence the above corollary gives us a way to produce nonsingular curves lying on a nonsingular cubic surface in $\mathbb{P}^3$. For example taking $a = 7$, $b_1 = b_2 = 3$, $b_3 = b_4 = b_5 = b_6 = 2$ we obtain a nonsingular irreducible curve $\mathcal{C} \sim al - \sum b_i e_i$, which has degree 7 and genus 5.

2.6 Conclusion

Having reached the end of our affords, we present the conjecture of Halphen who misleadingly stated as a theorem:

Conjecture (Halphen) : All possible values of the degree-genus pairs (d,g) of irreducible nonsingular curves Y in $\mathbb{P}^3$ are determined as follows;

(a) Plane curves, for any $d > 0$, with

$$g = \frac{1}{2}(d-1)(d-2).$$

(b) Curves on quadric surfaces, for any $a, b > 0$, with

$$d = a + b,$$

$$g = (a-1)(b-1).$$

(c) If Y does not lie on a plane or a quadric surface, then

$$g \leq \frac{1}{6}d(d-3) \ + \ 1.$$

(d) For any given $d > 0$, and g with $0 \leq g \leq \frac{1}{6}d(d-3) \ + \ 1$, there is a curve $Y \subseteq \mathbb{P}^3$ with degree d, and genus g.

As we have stated, Halphen claimed to have constructed the curves whose existence is mentioned in part (d) of the above conjecture on cubic surfaces. But later it has been found out that some gaps exist for the genus prescribed in part (d) of Halphen's statement, even allowing singularity on a cubic surface. In this way, Halphen's claim has been understood to be wrong. The difficulty lying in Halphen's theorem is the existence of curves with prescribed genus g as in part (d) of the theorem, which relied on general position arguments when it appeared publicly. Gruson, and Peskine realized this mistake as they tried to understand Halphen's work better, and corrected the fault by their work. The existence is the consequence of the two results stated below:

Proposition 2.6.1 *For any $d > 0$ and any $g \in \mathbb{Z}$ satisfying*

$$\frac{d^{\frac{3}{2}}}{\sqrt{3}} - d + 1 < g \leq \frac{1}{6}d(d-3) + 1,$$

there exists an irreducible nonsingular curve $\mathcal{C}$ of degree d and genus g on a nonsingular cubic surface in $\mathbb{P}^3$.

Proposition 2.6.2 *For any $d > 0$ and any $g \in \mathbb{Z}$ satisfying*

$$0 \leq g \leq \frac{1}{8}(d-1)^2,$$

there is an irreducible nonsingular curve $\mathcal{C}$ of degree d and genus g on a (singular) rational quartic surface in $\mathbb{P}^3$.

For the proofs and detailed explanation of the above stated propositions, we refer to Gruson and Peskine's research paper [9], [10], or a detailed exposition of their work, e.g. [17]. In their attempt to prove the two results stated above, Gruson and Peskine followed a very different route than the one followed by Halphen. They used the theory of integral quadratic forms. Some of their methods and ideas is exposed in Hartshorne's paper [15]. Since there occured many attempts to set a bound for the genus of a space curve of given degree, the following is the complete answer.

Theorem 2.6.1 *(Castelnuovo) Let X be a curve of degree d and genus g in $\mathbb{P}^3$, which is not contained in any plane. Then $d \geq 3$, and*

$$g \leq \begin{cases} \frac{1}{4}d^2 - d + 1 & \text{if } d \text{ is even} \\ \frac{1}{4}(d^2 - 1) - d + 1 & \text{if } d \text{ is odd} \end{cases}$$

Furthermore, the equality is attained for every $d \geq 3$, and every curve for which equality holds lies on a quadric surface.

Proof: We refer to [13] pp. 351-352. □

And as the last word for possible degree-genus pairs $(d, g) \in \mathbb{Z} \times \mathbb{Z}$ we present the theorem of S. Mori who closed the degree-genus pair problem in 1984 by the following theorem, which at the end by the above stated results proves the existence statement expressed in part (d) of Halphen's conjecture on nonsingular quartics:

Theorem 2.6.2 *(MORI) (in particular $k = \mathbb{C}$) Let k be an algebraically closed field with* char $k = 0$*. Given any $d > 0$ and $g \geq 0$ integers, there exists an* *irreducible nonsingular curve* $\mathcal{C}$ *of degree* d *and genus* g *on a nonsingular quartic surface* X *in* $\mathbb{P}^3$ *(depending on* $\mathcal{C}$*), if and only if* $d > 0$ *and either*

(a) $g = \frac{1}{8}d^2 + 1$ *or*

(b) $g < \frac{1}{8}d^2$ *and* $(d, g) \neq (5, 3)$.

For the proof, we refer to [21].

Chapter 3

Problems for Future Research

Problem-1:

The knowledge which assures that a nonsingular abstract curve can be embedded in some $\mathbb{P}^N$, and that it can be projected down till $\mathbb{P}^3$ without altering genus and nonsingularity was the motivation for starting the attempts to classify nonsingular projective curves in $\mathbb{P}^3$ and the possible degree-genus pairs $(d, g) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. We also know that any nonsingular curve can be projected birationally to a curve in $\mathbb{P}^2$ with singularities at most as nodes, and that if projection from a point defines a birational morphism $\varphi : X \subseteq \mathbb{P}^3 \to \mathbb{P}^2$, then the resulting image $\varphi(X)$ must be singular. Consider singular irreducible curves $\mathcal{C}$ of degree d in $\mathbb{P}^2$ containing r nodes and k cusps. Visually, a *node* looks like $xy = 0$ and a cusp look like $y^2 = x^3$. What are the possible triples (d, r, k) which can occur? The answer for small d is (cf. [31] pg. 57)

Degree(d)	(d, r, k)
1	$(1, 0, 0)$
2	$(2, 0, 0)$
3	$(3, 1, 0), (3, 0, 1)$
4	$(4, 3, 0), \ldots$

In general, this is an open problem. The complete answer is probably known up to degree 10, or an upper bound around. One natural constraint comes from the fact that if $\widetilde{\mathcal{C}}$ is the normalization of $\mathcal{C}$ then

$$g(\widetilde{\mathcal{C}}) = \frac{1}{2}(d-1)(d-2) - r - k \geq 0$$

So $r + k \leq \frac{1}{2}(d-1)(d-2)$. But it is a delicate question whether we can impose lower and upper bounds for r, and k depending upon the degree d as it has been given by the work of Halphen, Gruson, Peskine, and Mori for the nonsingular case in $\mathbb{P}^3$.

Problem-2: Since Halphen and Noether's trial towards the classification problem mathematical research has concentrated on the existence of curves on suitable surfaces on which the curve lies. In connection with this viewpoint the following invariants have been defined (cf. [15]).

$d =$ degree of $\mathcal{C}$,
$g =$ genus of $\mathcal{C}$,
$s =$ least degree of a surface containing $\mathcal{C}$,
$e =$ least integer for which $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(e)) \neq 0$.

One could ask what possible $4-$uples (d, g, s, e) occur for curves in $\mathbb{P}^3$, but this is a more difficult problem than finding only the admissible degree-genus pairs (d, g).

Problem-3: Let $\mathcal{C}$ be a curve in $\mathbb{P}^n$ for some $n \in \mathbb{N}$, and $\widetilde{\mathcal{C}}$ be the normalization of the curve $\mathcal{C}$. Then there is a morphism $f : \widetilde{\mathcal{C}} \to \mathcal{C}$, and there is an exact sequence

$$0 \to \mathcal{O}_{\mathcal{C}} \to f_* \mathcal{O}_{\widetilde{\mathcal{C}}} \to \sum_{P \in \mathcal{C}} \widetilde{\mathcal{O}_P}/\mathcal{O}_P \to 0,$$

where $\widetilde{\mathcal{O}_P}$ is the integral closure of $\mathcal{O}_P$. For each $P \in \mathcal{C}$, let $\delta_P = \text{length}(\widetilde{\mathcal{O}_P}/\mathcal{O}_P)$. Then we have the equality

$$p_a(\mathcal{C}) = p_a(\widetilde{\mathcal{C}}) + \sum_{P \in \mathcal{C}} \delta_P.$$

It has been shown by Max Noether that for a plane curve $\mathcal{C} \subseteq \mathbb{P}^2$, if the curve $\mathcal{C}$ has singularities at points $P_1, \ldots, P_m$ with multiplicities v_i at each P_i respectively then the sum of the delta invariants $\sum_{i=1}^{m} \delta_{P_i}$ is given by $\sum_{i=1}^{m} \delta_{P_i} = \sum_{i=1}^{m} \frac{1}{2} v_i(v_i - 1)$. Then a natural question is whether or not the normalization of $\mathcal{C}$ can be characterized by knowing the corresponding delta invariants δ_P at each singular point $P \in \mathcal{C}$.

Problem-4: (cf. [15]) For a nonsingular curve $\mathcal{C} \subseteq \mathbb{P}^3$ what is the largest integer $e \in \mathbb{Z}^+$ for which $H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(e)) \neq 0$. These integers behave in a semi-continuous manner in a flat family, so they can jump within some irreducible component of the Hilbert scheme $H(d, g)$.

Bibliography

[1] P.M. Cohn, "Algebra-Vol. 2", (John Wiley & Sons, Great Britain-Avon, 1989).

[2] P. Deligne, D. Mumford, *The irreducibility of the space curves of given genus*, Inst. Hautes Études Sci. Publ. Math. **36**, 75-109, 1969

[3] I. Dolgachev, "Introduction to Algebraic Geometry", notes for a class taught at the University of Michigan, available at `http://www.math.lsa.umich.edu/~idolga/lecturenotes.html`.

[4] D. Eisenbud, "Commutative Algebra with a Wiev Toward Algebraic Geometry", (Springer-Verlag, New York, 1995).

[5] D. Eisenbud, "The Geometry of Syzygies: A second course in Commutative Algebra and Algebraic Geometry", notes originated from a class taught at the Institut Poincaré in Paris, available at `http://www.msri.org/people/staff/de/ready.pdf`.

[6] R. Friedman, "Algebraic Surfaces and Holomorphic Vector Bundles", (Springer-Verlag, New York, 1998).

[7] A. Gathmann, "Algebraic Geometry", notes for a one-semester course taught in the mathematics international program at the University of Kaiserslautern, available at `http://www.mathematik.uni-kl.de/~gathmann/pdf/261.pdf`.

[8] P. Griffiths, J. Harris, "Principles of Algebraic Geometry", (John Wiley & Sons, New York, 1978).

[9] L. Gruson, C. Peskine, *Genre des courbes de l'espace projectif*, Algebraic Geometry, Tromsϕ 1977, Lecture Notes in Math. **687**, Springer-Verlag, 31-59 (1978).

[10] L. Gruson, C. Peskine, *Genre des courbes de l'espace projectif II*, Ann. Sc. E.N.S. (4) **15**, 401-418, (1982).

[11] G. Halphen, *Mémoire sur la classification des courbes gauches algébriques*, J. Éc. Polyt. **52**, 1-200 (1882).

[12] J. Harris , "An introduction to the moduli space of curves", Mathematical Aspects of String Theory (San Diego, CA 1986), 285-312, Adv. Ser. Math. Phys. **1**, World Sci., 1987.

[13] R. Hartshorne, "Algebraic Geometry", (Springer-Verlag, New York, 1977).

[14] R. Hartshorne, *On the classification of algebraic space curves*, "Vector Bundles and Differential Equations", (Nice, 1979), ed. by A. Hirschowitz, (Birkhäuser, Boston, pp. 83-112, 1980).

[15] R. Hartshorne, *On the classification of algebraic space curves, II*, Proceedings of Symposia in Pure Mathematics **46**, 145-164 (1987).

[16] R. Hartshorne, *Classification of algebraic space curves, III*, Algebraic Geometry and Its Applications, edited by C.L. Bajaj, Springer-Verlag, 113-120 (1994).

[17] R. Hartshorne, *Genre des courses algébriques dans l'espace projectif*, Séminaire Bourbaki, **592**, 301-313 (1982).

[18] F. Kirwan, "Complex Algebraic Curves", (London Math. Society Student Texts **12**, Cambridge Univ. Press, 1992).

[19] S.L. Kleiman, D. Laksov, *Another proof of the existence of special divisors*, Acta Math., **132**, 163-176 (1974).

[20] K. Matsuki, "Introduction to the Mori Program", (Springer-Verlag, New York, 2002).

[21] S. Mori, *On degrees and genera of curves on smooth quartic surfaces in* $\mathbb{P}^3$, Nagoya Math. J. **96**, 127-132 (1984).

[22] D. Mumford, *Further pathologies in algebraic geometry*, Amer.J.Mathematics **84**, 642-648 (1962)

[23] D. Mumford, "Algebraic Geometry I-Complex Projective Curves", (Springer-Verlag, Berlin, Heidelberg, 1976)

[24] D. Mumford, "Lectures on Curves on an Algebraic Surface", Annals of Math. Studies **59**, (Princeton U. Press, Princeton, 1966).

[25] M. Nagata, *On the embedding problem of abstract varieties in projective space*, Mem. Coll. Sci. Kyoto (A) **30**, 71-82 (1956).

[26] M. Nagata, *A general theory of algebraic geometry over Dedekind domains*, Amer.J.of Math. **78**, 78-116 (1956).

[27] M. Nagata, *Imbedding of an abstract variety in a complete variety*, J. Math. Kyoto Univ. **2**, 1-10 (1962).

[28] M. Noether, "Zur Grungdlegung der Theorie der algebraischen Raumcurven", (Verl. König. Akad. Wiss., Berlin, 1883)

[29] I.R. Shafarevich, "Basic Algebraic Geometry I-Varieties in Projective Space", (Springer-Verlag, Berlin Heidelberg, 1994).

[30] I.R. Shafarevich, "Basic Algebraic Geometry II-Schemes and Complex Manifolds", (Springer-Verlag, Berlin Heidelberg, 1994).

[31] W. Stein, "Notes for Algebraic Geometry II", notes from a course taught by Robin Hartshorne at U.C. Berkeley, available at `http://modular.fas.harvard.edu/AG.html`

[32] O. Zariski, *The concept of a simple point on an abstract algebraic variety*, Trans. Amer. Math. Soc. **62**, 1-52 (1947).

[33] O. Zariski, "Collected Papers-Vol. I. Foundations of Algebraic Geometry and Resolution of Singularities", ed. H. Hironaka and D. Mumford, (M.I.T. Press, Cambridge, 1972).

[34] O. Zariski, P. Samuel, "Commutative Algebra-Vol. I, II", (Van Nostrand, Princeton, 1958, 1969).

[35] R. Vakil, "Introduction to Algebraic Geometry", notes for a class taught at M.I.T., available at `http://math.stanford.edu/~vakil/725/course.html`.

[36] R. Vakil, "Topics in Algebraic Geometry: Complex Algebraic Surfaces", notes for a class taught at Stanford University, available at `http://math.stanford.edu/~vakil/245/`.