LOGARITHMIC DIMENSION AND BASES IN WHITNEY SPACES

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By

Yasemin Şengül

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Alexander Goncharov (Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Mefharet Kocatepe

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Alexander Shumovsky

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Aurelian Gheondea

Approved for the Institute of Engineering and Science:

Prof. Dr. Mehmet B. Baray
Director of the Institute Engineering and Science

ABSTRACT

LOGARITHMIC DIMENSION AND BASES IN WHITNEY SPACES

Yasemin Şengül
M.S. in Mathematics
Supervisor: Assoc. Prof. Alexander Goncharov
September, 2006

In generalization of [3] we will give the formula for the logarithmic dimension of any Cantor-type set. We will demonstrate some applications of the logarithmic dimension in Potential Theory. We will construct a polynomial basis in $\mathcal{E}(K(\Lambda))$ when the logarithmic dimension of a Cantor-type set is smaller than 1. We will show that for any generalized Cantor-type set $K(\Lambda)$, the space $\mathcal{E}(K(\Lambda))$ possesses a Schauder basis. Locally elements of the basis are polynomials. The result generalizes theorems 1 and 2 in [12].

Keywords: logarithmic dimension, Whitney spaces, topological bases.

ÖZET

LOGARITMIK BOYUT VE WHITNEY UZAYLARINDA BAZLAR

Yasemin Şengül
Matematik, Yüksek Lisans
Tez Yöneticisi: Yrd. Doç. Alexander Goncharov
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[3]'ün genellemesi olarak, herhangi bir Cantor-tipi kümenin logaritmik boyutunun formülünü vereceğiz. Logaritmik boyut'un Potansiyel Teorisi'ndeki bazı uygulamalarını göstereceğiz. $\mathcal{E}(K(\Lambda))$ uzayında, logaritmik boyutu $1'$den küçük olan Cantor-tipi bir küme için polinom bir baz oluşturacağız. Herhangi bir genelleştirilmiş Cantor-tipi küme için, $\mathcal{E}(K(\Lambda))$ uzayının bir Schauder bazına sahip olduğunu götereceğiz. Bu bazın elemanları lokal olarak polinomdurlar. Sonucumuz [12]'deki 1. ve 2. teoremleri genellemektedir.

Anahtar sözcükler: logaritmik boyut, Whitney uzayları, topolojik bazlar.

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Contents

Chapter 1

Introduction

Logarithmic dimension was introduced by Arslan, Goncharov and Kocatepe in [3] about five years ago. In this paper, they gave a formula for the calculation of the logarithmic dimension for the so-called regular case of Cantor-type sets. Now, we will give its calculation for any Cantor-type set. Moreover, the importance of logarithmic dimension for the class $\mathcal{E}(K)$ of Whitney functions defined on generalized Cantor sets has been studied in the same paper. The three authors investigated the problem of geometric characterization of the extension property of K and the diametral dimension of the space $\mathcal{E}(K)$. We will show an application of logarithmic dimension to Potential Theory. By the help of results of Lindelöf, Carleson, Erdös, Gillis and others, we will show that 1 is the critical value of logarithmic dimension for a set to be polar. We will support our results by examples.

Another subject which we relate to logarithmic dimension will be the basis problem in Whitney spaces. Generalizing the case handled by Goncharov in [12], we will construct a polynomial basis in case the logarithmic dimension of a set is smaller than 1.

After that, we will show that for any generalized Cantor-type set $K(\Lambda)$ in the space $\mathcal{E}(K(\Lambda))$, there exists a basis consisting of local polynomials.

Chapter 2

Logarithmic Dimension

2.1 Haussdorff Dimension

Intuitively, the dimension of a set is the number of independent parameters needed to describe a point in the set. One mathematical concept which closely models this idea is that of topological dimension of a set. For example, a point in the plane is described by two independent parameters, so in this sense, the plane is two-dimensional. However, topological dimension behaves in quite unexpected ways on certain highly irregular sets such as Cantor set, which has topological dimension zero, but in some sense behaves as a higher dimensional space. Hausdorff dimension gives another approach to this idea. It has the advantage of being defined for any set, and is mathematically convenient as is based on measures. A major disadvantage is that in many cases it is hard to calculate. ([11])

2.1.1 Hausdorff Outer Measure

Let φ be an increasing, continuos function from $[0, \infty)$ to $[0, \infty)$ and assume further that $\varphi(0) = 0$. Let a compact set K in $\mathbb{R}^N$ be given. By $B(r_n)$ we denote an open ball with radius r_n. Having $K \subset \bigcup B(r_n)$ we define

$$\widetilde{\mu}(K, \varphi) = \inf \sum \varphi(r_n),$$

taken over all coverings of K as the *Hausdorff outer measure of compact set* K *with respect to the function* φ (see e.g. [1]). Clearly, $\widetilde{\mu}(K, \varphi)$ takes only finite values. The function $\widetilde{\mu}$ is not additive, so it is not a measure. To show the lack of additivity, let us take for example,

$$A = [0, 1] \ , \ B = [2, 3] \ , \text{and} \ \varphi(r) = \sqrt{r} \, .$$

Then $\widetilde{\mu}(A \bigcup B, \varepsilon) \neq \widetilde{\mu}(A, \varphi) + \widetilde{\mu}(B, \varphi)$ as $\widetilde{\mu}(A, \varphi) = 1/2$, $\widetilde{\mu}(B, \varphi) = 1/2$, and $\widetilde{\mu}(A \bigcup B, \varepsilon) = \sqrt{3}/2$.

2.1.2 Hausdorff Measure

Now, let us consider only $\varepsilon-$covers of K, that is, coverings of K by balls with radii less than given ε :

$$\mu_\varepsilon(K, \varphi) \ = \ \inf \Sigma \varphi(r_n),$$

where infimum is taken over all coverings $K \subset \bigcup B(r_n)$, $r_n < \varepsilon$. It is easy to see that the value $\mu_\varepsilon(K, \varphi)$ increases as $\varepsilon \downarrow 0$. Therefore there exists a limit

$$\lim_{\varepsilon \to 0} \mu_\varepsilon(K, \varphi) = \mu(K, \varphi), \tag{2.1}$$

a *Hausdorff measure of* K *with respect to* φ. We have $0 \leq \mu(K, \varphi) \leq +\infty$.

Note that an equivalent definition of Hausdorff measure is obtained if the infimum in the definition of the outer measure is taken over covers of K by convex sets rather than by arbitrary sets since any set lies in a convex set of the same diameter. Similarly, it is sometimes convenient to consider covers of open, or alternatively of closed, sets. ([9])

It may be shown that $\mu(\cdot, \varphi)$ is a Borel regular measure on $\mathbb{R}^n$, so in particular,

$$\mu\left(\bigcup_{i=1}^{\infty} K_i\right) \leq \sum_{i=1}^{\infty} \mu(K_i)$$

for all sets $K_1, K_2, \ldots$, with equality if the K_i are disjoint Borel sets. ([10])

The classical choice for φ is $\varphi_\lambda(r) = r^\lambda$, $\lambda > 0$. Furthermore, Hausdorff measure generalizes Lebesgue measures, so that $\mu(K, r)$ gives, up to coefficients, the 'length' of a set or curve K, and $\mu(K, r^2)$ gives the (normalized) 'area' of a region or surface, etc. ([9])

We often wish to consider the Hausdorff measure of the image of a set under a Lipschitz mapping. For a Lipschitz $f : K \to \mathbb{R}^n$ such that for some constant c

$$|\, f(x) - f(y)\,| \le c\,|x - y| \text{ for all } x, y \in K,$$

we have

$$\mu(f(K), \varphi_\lambda) \le c^\lambda \mu(K, \varphi_\lambda).$$

Similarly, if $f : K \to \mathbb{R}^m$ is *bi-Lipschitz*, so that for some $c_1, c_2 > 0$

$$c_1|x - y| \le |\, f(x) - f(y)\,| \le c_2|x - y| \text{ for all } x, y \in K,$$

then

$$c_1^\lambda \mu(K, \varphi_\lambda) \le \mu(f(K), \varphi_\lambda) \le c_2^\lambda \mu(K, \varphi_\lambda).$$

A special case of this is when f is a *similarity transformation* of ratio r, so $|\, f(x) - f(y)\,| = r\,|x - y|$ for all $x, y \in K$, in which case

$$\mu(f(K), \varphi_\lambda) = r^\lambda \mu(K, \varphi_\lambda).$$

This is the *scaling property* of Hausdorff measures, which generalizes the familiar scaling properties of length, area, volume, etc. ([10])

2.1.3 Hausdorff Dimension

Returning to equation (2.1), it is clear that for any given set E and $\varepsilon < 1$, $\mu(E, \varphi_\lambda)$ is non-increasing with λ. One can show (see e.g.[11], p.28) that there exists a critical value $\lambda_0 = dimE$, $0 \le \lambda_0 \le \infty$, at which $\mu(E, r^\lambda)$ 'jumps' from

∞ to $0, i.e,$

$$\mu(E\,, r^{\lambda}) = \begin{cases} \infty, & \text{if } \lambda < \lambda_0 \\ 0, & \text{if } \lambda > \lambda_0 \end{cases}$$

This critical value, λ_0, is called the *Hausdorff dimension* of E. In general $\mu(E\,, r^{\lambda_0})$ can take any value from $[0, +\infty]$. The Hausdorff dimension of the ball in the Euclidean space $\mathbb{R}^n$ coincides with the dimension n of the space. Also $\lambda_0(line) = 1, \lambda_0(plane) = 2$, etc. A non-trivial example is the Cantor-ternary set, which has Hausdorff dimension $\lambda_0 = \frac{\log 2}{\log 3}$. Basic properties of Hausdorff dimension are (see e.g.[10]);

Monotonicity: If $E_1 \subset E_2$ then $dim E_1 \leq dim E_2$.

Finite sets: If E is finite, then $dim E = 0$.

Open sets: If E is a (non-empty) open subset of $\mathbb{R}^n$, then $dim E = n$.

Smooth manifolds: If E is a smooth $m-$dimensional manifold in $\mathbb{R}^n$, then $dim E = m$.

Lipschitz mappings: If $f : E \rightarrow \mathbb{R}^m$is Lipschitz, then $dim\, f(E) \leq dim E$.

Bi-Lipschitz invariance: If $f : E \rightarrow f(E)$ is bi-Lipschitz, then $dim\, f(E) = dim E$.

Geometric invariance: If f is a similarity or affine transformation, then $dim\, f(E) = dim E$ (this is a special case of bi-Lipschitz invariance).

2.2 Logarithmic Dimension

Here and in what follows, log denotes the natural logarithm. Logarithmic dimension was introduced in [3] as the following generalization of the Hausdorff dimension: take the function $\psi(r) = \frac{1}{\log \frac{1}{r}}$, $0 < r < 1$, corresponding to the logarithmic measure; then for any compact set K there exists a critical value $\lambda_0 = \lambda_0(K) \in [0, \infty]$ (called the *logarithmic dimension of* K) such that for $\lambda < \lambda_0$

the $\psi^\lambda-$measure of K is ∞ , for $\lambda > \lambda_0$ it is zero.

Let us give some properties of logarithmic dimension.

Monotonicity: If $K_1 \subset K_2$ then $\lambda_0(K_1) \leq \lambda_0(K_2)$.

Finite sets: If K is finite then $\lambda_0(K) = 0$.

Intervals: If K contains an interval then $\lambda_0(K) = \infty$. *Proof*: Without loss of generality let $K = [0, 1]$. Let us fix any covering $\bigcup U_\lambda$ of K by open intervals. We will show that

$$\mu([0,1], \psi^N) \; = \; \lim_{\varepsilon \to 0} \mu_\varepsilon([0,1], \; \psi^N) \; = \; \infty, \; \forall N.$$

As $[0, 1]$ is compact, there exists a finite covering, i.e, $[0, 1] \subset \bigcup_1^M U_i$. Let $k_m = |\{\, i : \frac{1}{m+1} \; \leq 2 \cdot |U_i| \; < \frac{1}{m}\}|$. Then clearly,

$$[0,1] \; < \; \sum_1^M |U_i| \; < \; \sum_{m=n}^{n_1} \frac{k_m}{m}$$

for some n and n_1. So,

$$\sum_i \psi^N(|U_i|) \; \geq \sum_{m=n}^{n_1} \frac{k_m}{\left(\log \frac{1}{m+1}\right)^N} \; = \; \sum_{m=n}^{n_1} \frac{k_m}{m} \cdot \frac{m}{\left(\log \frac{1}{m+1}\right)^N}.$$

If $\varepsilon \; \to \; 0$ and $|U_i| \; < \; \varepsilon$, then n and consequently m tend to infinity. As $\frac{m}{\left(\log \frac{1}{m+1}\right)^N} \; \to \infty$ and $\Sigma \, \frac{k_m}{m} \; > \; 1$, the result follows.

□

Logarithmic dimension takes not infimum values only for rather rarefied compact sets. Even, the logartihmic dimension of the classical Cantor set is ∞. It can be proved in the same way as above.

2.3 Generalized Cantor-type sets

We consider the following generalization of the Cantor ternary set as in [3]. Let $(l_n)_{n=1}^\infty$ be a sequence of positive numbers and $(N_n)_{n=1}^\infty$ be a sequence of integers,

$N_n \geq 2$ for all n. Let $\alpha_1 = 1$ and for $n \geq 2$ let α_n satisfy $l_n = l_{n-1}^{\alpha_n}$. Then $K = K_{(N_n)}^{(\alpha_n)} = \bigcap_{n=0}^{\infty} E_n$, where $E_0 = I_{0,1} = [0,1]$ and $E_n, n \geq 1$ is a union of $N_1 N_2 \cdots N_n$ disjoint closed intervals $I_{n,k}$ of length l_n and E_{n+1} is obtained by replacing each interval by N_{n+1} disjoint subintervals $I_{n+1,j}$ of length l_{n+1} with $N_{n+1} - 1$ equal gaps of length h_{n+1}. The intervals $I_{n,k}$ that make up the set E_n are called *basic intervals*. The set K is well-defined if for all n we have $l_{n-1} > N_n l_n$ with $l_0 = 1$. Then $h_n = \frac{l_{n-1} - N_n l_n}{N_n - 1}$ is a gap between two adjacent intervals of the same length. We will denote by $K_N^{(\alpha)}$ the case when $N_n = N$ and $\alpha_n = \alpha, \forall n$.

2.4 Relation to Potential Theory

The value $\lambda_0 = 1$ is critical in Potential Theory: if $\lambda_0(K) < 1$, then the logarithmic measure of K is 0 and the set K is exceptional, meaning $c(K) = 0$, where $c(K)$ denotes the logarithmic capacity of the set K. Let us recall the notion of capacity in more details.

Let α be a point set consisting of finitely many boundary arcs of the region G. There exists one and only one bounded harmonic function in G that takes the value 1 at every interior point of α, while vanishing at every point of the complementary portion α' of the boundary $\partial G = \alpha + \alpha'$. We call the value taken by this function at an interior point z of G *the harmonic measure of the arc* α *with respect to* G *at the point* z; we denote it by $\omega(z, \alpha, G)$. The harmonic measure ω is constant and equal to 1 if α is the entire boundary ∂G; otherwise, ω varies in G between 0 and 1. If α and β are two disjoint arcs, it is always true that

$$\omega(z, \alpha) + \omega(z, \beta) = \omega(z, \gamma),$$

where γ stands for the union $\gamma = \alpha + \beta$. Harmonic measure is thus an *additive* function of the measured boundary arcs. ([24])

Now, we fix our attention on a region $G \subset \overline{\mathbb{C}}$ that contains the point $z = \infty$ and that is bounded by finitely many Jordan arcs. Let K be the compact set complementary to G in $\overline{\mathbb{C}}$. The associated Green's function has an expansion

$$g_K(z) = g(\overline{\mathbb{C}} \setminus G, z, \infty) = \log |z| + u(z)$$

near the pole $z = \infty$, where $u(z)$ is harmonic at $z = \infty$ and assumes a finite value

$$\gamma = u(\infty)\,.$$

The value γ is called the *Robin constant for* G, and the quantity

$$c(K) = e^{-\gamma}$$

is referred to as *the (logarithmic) capacity of the compact set* $\overline{\mathbb{C}} \setminus G$. ([24])

The idea of the capacity of a set was originally developed for treating electrostatic problems. The theory was extended to more general laws of attraction in the branch of mathematics known as potential theory, much of the early work being formulated in the famous thesis of Frostman (1935). (For more recent accounts see Taylor (1961), Carleson (1967), Hayman and Kennedy (1976) or Hille (1973)). It turns out that the Hausdorff dimension and the capacity of a set are related, and it is sometimes more convenient to use the latter concept when studying dimensional properties. ([9])

In Potential Theory the following question is of great importance: when the set K is polar (that is $c(K) = 0$) or exceptional (that is $\omega(\cdot, K, G) = 0$). The following results are known (see e.g.[24]);

Proposition 1 *A closed point set of logarithmic measure zero is also of harmonic measure zero (pg.147).*

Proposition 2 *A set of zero harmonic measure is always of capacity zero, and conversely (i.e,* $\omega(z, K, G) = 0 \Leftrightarrow c(K) = 0$*)(pg.123).*

Proposition 3 *(Lindelöf)* $\mu_1(K) = 0 \Rightarrow c(K) = 0$, *where* $\mu_1(K)$ *=logarithmic measure of the compact set* K.

Later in 1937, the Lindelöf Theorem was strengthened as follows:

Proposition 4 *(Erdös-Gillis)* $\mu_1(K) < \infty \Rightarrow c(K) = 0$.

As an example, let us consider the case of Cantor-type sets. By Carleson [5] (see also [6]);

Proposition 5 $c(K_2^{(\alpha_n)}) = 0 \Leftrightarrow \sum_1^\infty \frac{A_n}{2^n} = \infty$, *where* $A_n = \alpha_1 \alpha_2 \ldots \alpha_n$.

From this proposition we can derive the following:

Lemma 1 *Assume* $\lambda_0 = \lambda_0(K_2^{(\alpha_n)}) \neq 1$. *Then* $c(K_2^{(\alpha_n)}) = 0 \Leftrightarrow \lambda_0 < 1$.

Proof: ($\Rightarrow$) We use here the formula $\lambda_0(K_2^{(\alpha_n)}) = \liminf_n \frac{\log 2^n}{\log A_n}$, that will be proved later (2.6, T.1). Also,

$$\mu_1(K_2^{(\alpha_n)}) = \lim_{\varepsilon \to 0} \mu_\varepsilon(K_2^{(\alpha_n)}, \varphi) = \lim_{n \to \infty} 2^n \cdot \frac{1}{\log \frac{1}{l_n}} = \lim_{n \to \infty} \frac{2^n}{A_n}.$$

Here without loss of generality we take $l_1 = \exp^{-1}$. Suppose that $\lambda_0 > 1$, i.e, $\liminf_n \frac{\log 2^n}{\log A_n} > 1$, say $\liminf_n \frac{\log 2^n}{\log A_n} = 1 + \varepsilon$. Then, $\forall \delta, \exists n_0 : \forall n \geq n_0$, $\left| \frac{\log 2^n}{\log A_n} - (1+\varepsilon) \right| < \delta$. This means

$$1 + \varepsilon - \delta < \frac{\log 2^n}{\log A_n} < 1 + \varepsilon + \delta,$$

equivalently by defining new constants γ and η we can write

$$1 + \gamma < \frac{\log 2^n}{\log A_n} < 1 + \eta.$$

Then, considering the first inequality;

$$\begin{aligned} \frac{\log 2^n}{\log A_n} > 1 + \gamma \quad &\Leftrightarrow \quad \log A_n < \frac{1}{1+\gamma} \cdot \log 2^n \\ &\Leftrightarrow \quad \log A_n < \log 2^{n(1-\zeta)} \\ &\Leftrightarrow \quad A_n < 2^{n(1-\zeta)} \end{aligned}$$

Hence,

$$\frac{A_n}{2^n} < \frac{1}{2^{n\zeta}} = \left(\frac{1}{2^\zeta} \right)^n.$$

As $1 < 2^\zeta$, $\forall \zeta$, we have

$$\sum \left(\frac{1}{2^\zeta} \right)^n < \infty,$$

and

$$\sum \frac{A_n}{2^n} < \sum \left(\frac{1}{2^\zeta}\right)^n < \infty.$$

Now,by Proposition 5, $c(K_2^{(\alpha_n)}) \neq 0$, which is a contradiction.

($\Leftarrow$) If $\lambda_0 < 1$, then $\exists \varepsilon : \mu(K, \psi^{1-\varepsilon}) = 0$. As $\mu(K, \psi) = \mu_1(K, \psi)$ by definition, and

$$\sum_i \psi^{1-\varepsilon}(r_i) > \sum_i \psi(r_i)$$

we have $\mu_1(K, \psi) = 0$. Now, the result follows from Proposition 3. □

Now,we can show by examples that the inverse implications in Propositions 3 and 4 do not hold. Take, for example,

$$A_n = \frac{2^n}{n}, \text{ i.e, } \alpha_2 = 2, \, \alpha_n = 2\frac{n-1}{n}, n \geq 3.$$

Then, by Proposition 5, $c(K_2^{(\alpha_n)}) = 0$ and $\mu_1(K_2^{(\alpha_n)}) = \liminf_n \frac{2^n}{A_n} = \infty$.

Also for$K_2^{(2)}$ we get $c(K_2^{(2)}) = 0$ and $\lambda_0(K_2^{(2)}) = 1$. Moreover, taking

$$A_n = \frac{2^n}{n^2}, \text{ i.e, } \alpha_{n+1} = 2\left(\frac{n}{n+1}\right)^2,$$

we get, $\lambda_0(K_2^{(\alpha_n)}) = 1$. But here, $c(K_2^{(\alpha_n)}) > 0$.

Thus for the case $\lambda_0 = 1$ we can obtain both kind of sets (polar and nonpolar). As a result we can conclude that the value $\lambda_0 = 1$ is a critical value in Potential Theory.

2.5 Logarithmic Dimension of $K_{(N_n)}^{(\alpha_n)}$ in general case

We say that the Cantor-type set $K_{(N_n)}^{(\alpha_n)}$ is regular if there exists the limit $\lim_n \frac{\log N_n}{\log \alpha_n}$. The logarithmic dimension of regular Cantor-type sets was given in [3], proposition 1:

Suppose that for $K^{(\alpha_n)}_{(N_n)}$ *the limit* $\lambda_0 = \lim_n \lambda_n$*, where* $\lambda_n = \frac{\log N_n}{\log \alpha_n}$*, exists in the set of extended real numbers. Then* λ_0 *is the logarithmic dimension of* K*. In particular* $\lambda_0(K^{(\alpha)}_N) = \frac{\log N}{\log \alpha}$.

Now, we can present our result for the logarithmic dimension of any generalized Cantor-type set.

Theorem 1 *For any generalized Cantor-type set* $K = K^{(\alpha_n)}_{(N_n)}$*, we have*

$$\lambda_0(K^{(\alpha_n)}_{(N_n)}) = \liminf_n \frac{\log(N_1 N_2 \cdots N_n)}{\log(\alpha_1 \alpha_2 \cdots \alpha_n)}$$

Proof: Here φ denotes the function $\varphi(r) = \frac{1}{\log \frac{1}{r}},\ r > 0$. Define $\lambda_n = \frac{\log(N_1 N_2 \cdots N_n)}{\log(\alpha_1 \alpha_2 \cdots \alpha_n)}$, for $n \geq 2$, so that $\lambda_0 = \liminf_n \lambda_n$. Clear that by definition of λ_n we have

$$(\alpha_1 \alpha_2 \cdots \alpha_n)^{\lambda_n} = N_1 N_2 \cdots N_n. \tag{2.2}$$

Let us consider two possible cases.

i) $\lambda_0 < \infty$: We need to show that, $\forall\, \lambda > \lambda_0$, we have $\mu(K, \varphi^\lambda) = 0$ and $\forall\, \lambda < \lambda_0$, we have $\mu(K, \varphi^\lambda) = \infty$.

Take $\forall\, \lambda < \lambda_0$. By definition of μ, for $\varepsilon > 0$ there exists a finite covering $\bigcup_{i=1}^M U_i$ of K by open intervals U_i, diam $U_i = 2r_i < 2\varepsilon$ such that $\sum \varphi^\lambda(r_i) \leq \mu_\varepsilon(K, \varphi^\lambda) + 1$. For each r_i fix $n = n(i) \in \mathbb{N}$ with $l_n \leq r_i \leq l_{n-1}$. Let $n_0 = \min_{i \leq M} n(i)$, $n_1 = \max_{i \leq M} n(i)$. To simplify calculations we set $l_1 = 1/e$. Then

$$\varphi^\lambda(r_i) \geq \varphi^\lambda(l_n) = (\alpha_1 \alpha_2 \cdots \alpha_n)^{-\lambda}. \tag{2.3}$$

As by definition, $\lambda_0 = \liminf_n \lambda_n$, we can say that $\exists n = n_\lambda$ such that $\lambda < \lambda_n$, $\forall n \geq n_\lambda$. When $\varepsilon \downarrow 0$ then clearly $n_0 \to \infty$. Therefore we can take ε so small that $\lambda_n > \lambda$ for $n \geq n_0 \geq n_\lambda$. Now for $n_0 \leq n \leq n_1$ we get

$$\begin{aligned}
(\alpha_1 \alpha_2 \cdots \alpha_n)^\lambda &= (\alpha_1 \alpha_2 \cdots \alpha_{n_0 - 1})^\lambda \cdot (\alpha_{n_0}^\lambda \cdots \alpha_n^\lambda) \\
&\leq (\alpha_1 \alpha_2 \cdots \alpha_{n_0 - 1})^\lambda \cdot (\alpha_{n_0}^{\lambda_n} \cdots \alpha_n^{\lambda_n}) \\
&= (\alpha_1 \alpha_2 \cdots \alpha_{n_0 - 1})^\lambda \cdot \frac{N_1 \cdots N_n}{(\alpha_1 \alpha_2 \cdots \alpha_{n_0 - 1})^{\lambda_n}} \\
&= (\alpha_1 \alpha_2 \cdots \alpha_{n_0 - 1})^{\lambda - \lambda_n} \cdot N_1 \cdots N_n.
\end{aligned} \tag{2.4}$$

We decompose the sum $\sum \varphi^\lambda(r_i)$ into two parts. Let $\sum'$ be the sum over all i such that $l_n \le r_i \le \frac{l_{n-1}}{N_n}$, and $\sum''$ be the sum over the remaining i's. Since $\frac{l_{n-1}}{N_n} < l_n + h_n$, for any i in the sum $\sum'$, the interval U_i can intersect at most two basic intervals of E_n. By construction, it can intersect at most $2N_{n+1}$ basic intervals of $E_{n+1}; \cdots ; 2N_{n+1}\cdots N_{n_1}$ basic intervals of E_{n_1}. Then, by (2.2) and (2.3),

$$\begin{aligned} 2N_{n+1}\cdots N_{n_1} &\le 2N_{n+1}\cdots N_{n_1}\cdot(\alpha_1\alpha_2\,\cdots\,\alpha_n)^\lambda\cdot\varphi^\lambda(r_i) \\ &\le 2N_1\cdots N_{n_1}(\alpha_1\alpha_2\,\cdots\,\alpha_{n_0-1})^{\lambda-\lambda_n}\cdot\varphi^\lambda(r_i). \end{aligned}$$

Therefore any interval U_i, corresponding to the sum $\sum'$ can intersect at most $2N_1\cdots N_{n_1}(\alpha_1\alpha_2\,\cdots\,\alpha_{n_0-1})^{\lambda-\lambda_n}\cdot\varphi^\lambda(r_i)$ intervals of E_{n_1}.

For i in the second sum $\sum''$, fix j, $j = 1, 2, \ldots, N_n - 1$, such that $\frac{j}{N_n}l_{n-1} \le r_i < \frac{j+1}{N_n}l_{n-1}$. Then the interval U_i can intersect at most $j+2$ basic intervals of E_n and thus $(j+1)N_{n+1}\cdots N_{n_1}$ basic intervals of E_{n_1}. Here

$$\varphi^\lambda(r_i) \ge \varphi^\lambda\left(\frac{j+1}{N_n}l_{n-1}\right) \ge \left(\alpha_1\alpha_2\,\cdots\,\alpha_{n-1} + \log\frac{N_n}{j}\right)^{-\lambda}.$$

If $\log\frac{N_n}{j} \ge \alpha_1\alpha_2\,\cdots\,\alpha_{n-1}$, then $1 \le 2^\lambda\,\log^\lambda(\frac{N_n}{j})\varphi^\lambda(r_i) \le C'_\lambda\,\frac{N_n}{j}\varphi^\lambda(r_i)$. Therefore

$$\begin{aligned} (j+2)N_{n+1}\cdots\, N_{n_1} &\le C''_\lambda N_n N_{n+1}\cdots N_{n_1}\varphi^\lambda(r_i) \\ &\le C''_\lambda N_{n_0}\cdots N_{n_1}\cdot(\alpha_1\alpha_2\,\cdots\,\alpha_{n_0-1})^\lambda\varphi^\lambda(r_i). \end{aligned}$$

On the other hand, if $\log\frac{N_n}{j} < \alpha_1\alpha_2\,\cdots\,\alpha_{n-1}$, then $1 \le 2^\lambda(\alpha_1\alpha_2\,\cdots\,\alpha_{n-1})\varphi^\lambda(r_i)$, therefore

$$\begin{aligned} (j+2)N_{n+1}\cdots\, N_{n_1} &\le (N_n+1)N_{n+1}\cdots N_{n_1}(\alpha_1\alpha_2\,\cdots\,\alpha_{n-1})^\lambda 2^\lambda\varphi^\lambda(r_i) \\ &\le 2^{\lambda+1}N_{n_0}\cdots N_{n_1}(\alpha_1\alpha_2\,\cdots\,\alpha_{n_0-1})^\lambda\varphi^\lambda(r_i). \end{aligned}$$

Now,we have

$$\frac{N_1\cdots N_{n_1}}{(\alpha_1\alpha_2\,\cdots\,\alpha_{n_0-1})^{\lambda_n}} < N_{n_0}\cdots N_{n_1}$$

because $(\alpha_1\alpha_2\,\cdots\,\alpha_{n_0-1})^{\lambda_{n_0-1}} < (\alpha_1\alpha_2\,\cdots\,\alpha_{n_0-1})^{\lambda_n}$ which implies $N_1\cdots N_{n_0-1} < (\alpha_1\alpha_2\,\cdots\,\alpha_{n_0-1})^{\lambda_n}$.

Thus any interval U_i, $i \leq M$ can intersect at most

$$C_\lambda N_{n_0} \cdots N_{n_1} (\alpha_1 \alpha_2 \, \cdots \, \alpha_{n_0 - 1})^\lambda \varphi^\lambda (r_i)$$

basic intervals of E_{n_1}. Here $C_\lambda = \max\{C''_\lambda, 2^{\lambda+1}\}$. Since the covering $\bigcup U_i$ intersects all basic intervals of E_{n_1}, we have

$$N_1 \cdots N_{n_1} \leq \, C_\lambda N_{n_0} \cdots N_{n_1} (\alpha_1 \alpha_2 \, \cdots \, \alpha_{n_0 - 1})^\lambda \sum \varphi^\lambda (r_i)$$

and so

$$\sum \varphi^\lambda (r_i) \;\; \geq \;\; C_\lambda^{-1} (\alpha_1 \alpha_2 \cdots \alpha_{n-1} \alpha_{n+1} \cdots \alpha_{n_0 - 1})^{\lambda_{n_0 - 1} - \lambda}$$

Here λ_n and λ are distant: for large n we get $\lambda_n - \lambda > \frac{\lambda_0 - \lambda}{2}$. This bound implies that the sum of the type $\sum \varphi^\lambda (r_i)$ must be arbitrarily large for small enough ε, that is $\mu(K\,, \varphi^\lambda) = \infty$, because even when $(\alpha_1 \alpha_2 \cdots \alpha_n) \to 1$, we have

$$\frac{N_1 \cdots N_{n_0 - 1}}{(\alpha_1 \alpha_2 \cdots \alpha_n \cdots \alpha_{n_0 - 1})^\lambda} \;\; \geq \;\; \frac{2^{n_0 - 1}}{(\alpha_1 \alpha_2 \cdots \alpha_n \cdots \alpha_{n_0 - 1})^\lambda} \to \infty, \text{as} \;\; n_0 \to \infty.$$

Now take $\forall \, \lambda \; > \; \lambda_0$. By definition of λ_n, we have $\lambda \; > \; \liminf_n \lambda_n$. Here, $\exists \, n_k \uparrow \infty \; : \; \lambda_0 \; = \; \lim_k \frac{\log(N_1 \, N_2 \, \cdots \, N_{n_k})}{\log \, (\alpha_1 \alpha_2 \, \cdots \, \alpha_{n_k})}$ and hence $\lambda > \lambda_{n_k}$ for large enough k. Then we have,

$$\begin{aligned}
\mu(K\,, \varphi^\lambda) \;\; &\leq \;\; \liminf_n \, (N_1 \, N_2 \, \cdots \, N_n) \, \varphi^\lambda (l_n) \\
&= \;\; \liminf_n \, \frac{N_1 \, N_2 \, \cdots \, N_n}{(\alpha_1 \alpha_2 \, \cdots \, \alpha_n)^\lambda} \\
&= \;\; \liminf_n \, (\alpha_1 \alpha_2 \, \cdots \, \alpha_n)^{\lambda_n - \lambda} \\
&\leq \;\; \liminf_k \, (\alpha_1 \alpha_2 \, \cdots \, \alpha_{n_k})^{\lambda_{n_k} - \lambda} \; = \; 0
\end{aligned}$$

by the fact that $(\alpha_1 \alpha_2 \, \cdots \, \alpha_n) \; \to \; \infty$, which we always have when $\lambda_0 \; < \; \infty$. Indeed, the sequence $(\alpha_1 \alpha_2 \, \cdots \, \alpha_n)_n$ is increasing. If $(\alpha_1 \alpha_2 \, \cdots \, \alpha_n) \; \leq \; M$ for some constant $M > 0$, then,

$$\lambda_0 \; = \; \liminf_n \, \frac{\log(N_1 \, N_2 \, \cdots \, N_n)}{\log(\alpha_1 \alpha_2 \, \cdots \, \alpha_n)} \geq \, \liminf_n \, \frac{\log 2^n}{\log M} \; = \; \infty$$

ii) $\lambda_0 = \infty$: We have $\liminf_n \frac{\log(N_1 N_2 \cdots N_n)}{\log(\alpha_1 \alpha_2 \cdots \alpha_n)} = \infty$. Then, $\lambda_n \to \infty$. We can repeat here all arguments from the case i) with the only difference: for any λ, (let wlog $\lambda \geq 1$) we take n_λ with $\lambda_n > 2\lambda$ for $n > n_\lambda$, and this ends the proof. $\square$

Chapter 3

Bases in $\mathcal{E}(K^{(\alpha_n)}_{(N_n)})$

3.1 The Basis Problem for Nuclear Fréchet Spaces

A *locally convex topological vector space* E has a topology that is defined by a family of seminorms, while in a Fréchet spaces the family of seminorms is countable.

A sequence $(e_j)_{j\in\mathbb{N}}$ in a locally convex space E over field $\mathbb{K}$ is called a *Schauder basis of* E, if for each $x \in E$, there is a uniquely determined sequence $(\xi_j(x))_{j\in\mathbb{N}}$ in $\mathbb{K}$, for which $x = \sum_{j=1}^{\infty} \xi_j(x)\, e_j$ is true. The maps $\xi_j : E \to \mathbb{K}$, $j \in \mathbb{N}$, are called the *coefficient functionals* of the Schauder basis $(e_j)_{j\in\mathbb{N}}$. They are linear by the uniqueness stipulations and continuous by the Banach-Schauder Theorem. (see e.g.[20])

A Schauder basis $(e_j)_{j\in\mathbb{N}}$ of E is called an *absolute basis*, if for each seminorm p on E there is a seminorm q on E and there is a $C > 0$ such that

$$\sum_{j\in\mathbb{N}} |\xi_j(x)| p(e_j) \leq C\, q(x) \quad \text{for all} \quad x \in E.$$

([21])

A *nuclear space* is a locally convex topological vector space E such that for any locally convex topological vector space F, the natural map from the projective to the injective tensor product of E and F is an isomorphism. ([17], see [18, 15-16] for details)

The Grothendieck problem of the existence of a basis in a nuclear Fréchet (NF) space was open for a long time. Only in 1974 the first example of NF space without basis was found by Zobin and Mityagin [30]. After this many other examples of nuclear spaces without basis were presented, but all of them are either artificial as in [4], [7], [23], [28] or nonmetrizable as in [8]. That is, till now, no 'natural' NF space of functions without basis has been found. This explains the interest to basis problem in concrete functional spaces.

Any Schauder basis in a NF space is absolute, therefore in order to construct a basis in such a space, it is enough to present a biorthogonal system satisfying the Dynin-Mityagin criterion:

Dynin-Mityagin criterion: *Let E be a nuclear Fréchet space and $\{e_n \in E,\, \xi_n \in E',\, n \in \mathbb{N}\}$ be a biorthogonal system such that the set of functionals $(\xi_n)_1^\infty$ is total over E. Let for every p there exist q and C such that for all n*

$$\| e_n\|_p \cdot |\, \xi_n|_{-q} \leq C.$$

Then the system $\{e_n\, ,\, \xi_n\}$ is an absolute basis in E.

Here, $|\cdot|_{-q}$ denotes the dual norm: for $\xi \in E'$ let $|\,\xi|_{-q} = \sup\{|\,\xi(f)|,\, \|f\|_q \leq 1\}$.

A matrix $A = (a_{j,k})_{j,k\in\mathbb{N}}$ of non-negative numbers is called a *Köthe matrix* if it satisfies the following conditions:

(1) For each $j \in \mathbb{N}$ there exists a $k \in \mathbb{N}$ with $a_{j,k} > 0$.

(2) $a_{j,k} \leq a_{j,k+1}$ for all $j, k \in \mathbb{N}$. ([21])

Given Köthe matrix A, by $K(A)$ we denote the *Köthe space*, that is the space of all sequences (x_j) such that for any $k \in \mathbb{N}$ the series

$$\sum_{j=1}^{\infty} |\, x_j|\, a_{jk}$$

converges. This is a Fréchet space with topology given by the seminorms

$$|\, x|_k \; = \; \sum_{j=1}^{\infty} |\, x_j|\, a_{jk}.$$

Any NF space with the basis $(e_j)_{j=1}^{\infty}$ is isomorphic to the Köthe space $K(A)$ where $a_{jk} \; = \; \|e_j\|_k$ (see e.g.[22]).

3.2 Whitney Spaces

Let U be an open subset of $\mathbb{R}^n$, and K a compact subset of U. Whitney's theorem asserts that a function F^0 defined in K is the restriction of a C^m ($m \in \mathbb{N}$) function in U provided that there exists a sequence $(F^k)_{|k| \leq m}$ of functions defined in K which satisfies certain conditions that arise naturally from Taylor's formula. ([2])

By a *jet* of order m on K, we mean a set of continuous functions $F = (F^k)_{|k| \leq m}$ on K. Here k denotes a multiindex $k = (k_1, \ldots, k_n) \in \mathbb{N}^n$. Let $J^m(K)$ be the vector space of jets of order m on K. We write

$$|\, F \,|_{\, m} \; = \; \sup |\, F^k(x) \; : \; x \in K, \; |k| \leq m$$

if $F(x) \; = \; F^0(x)$. ([2])

There is a linear mapping $J^m \; : \; \mathcal{E}^m(U) \; \rightarrow \; J^m(K)$which associates to each $f \in \mathcal{E}^m(U)$ the jet

$$J^m(f) \; = \; \left(\frac{\partial^{|k|} f}{\partial x^k} |\, K \right)_{|k| \leq m}$$

For each k with $|k| \leq m$, there is a linear mapping $D^k \; : \; J^m(x) \; \rightarrow \; J^{m-|k|}(K)$ defined by $D^k F \; = \; (F^{k+l})_{|l| \leq m-|k|}$. ([2])

If $a \in K$ and $F \in J^m(x)$, then the *Taylor polynomial of order* m *of* F *at* a is the polynomial

$$T_a^m F(x) = \sum_{|k| \le m} \frac{F^k(a)}{k!} (x-a)^k$$

of degree $\le m$. Here $k! = k_1! \dots k_n!$. We define $R_a^m F = F - J^m(T_a^m F)$, so that

$$(R_a^m F)^k(x) = F^k(x) - \sum_{|l| \le m-|k|} \frac{F^{k+l}(a)}{l!} (x-a)^l$$

if $|k| \le m$. ([2])

A jet $F \in J^m(K)$ is *a Whitney jet of class* C^m *on* K if for each $|k| \le m$

$$(R_x^m F)^k(y) = o(|x-y|^{m-|k|})$$

as $|x-y| \to 0$, $x, y \in K$. Let $\mathcal{E}^m(K) \subset J^m(K)$ be the subspace of Whitney fields of class C^m. $\mathcal{E}^m(K)$ is a Banach space with the norm

$$\| F \|_m = |F|_m + \sup \left\{ \frac{|(R_x^m F)^k(y)|}{|x-y|^{m-|k|}} : x, y \in K, x \neq y, |k| \le m \right\}.$$

([2])

Theorem (Whitney[16]) There is a continuous linear mapping

$$W : \mathcal{E}^m(K) \to \mathcal{E}^m(U)$$

such that $D^k W(F)(x) = F^k(x)$ if $F \in \mathcal{E}^m(K)$, $x \in K$, $|k| \le m$, and $W(F)/(U - K)$ is C^∞.

If $m = \infty$, then the spaces $\mathcal{E}(K) = \mathcal{E}^\infty(K)$ and $\mathcal{E}(U) = \mathcal{E}^\infty(U)$ can be defined as the corresponding projective limits. By Whitney Theorem, any $\mathcal{E}(K)$ has an extension to C^∞ function on U, but now, in general there is no continuous linear operator $W : \mathcal{E}(K) \to \mathcal{E}(U)$. We restrict our attention to the case when K is a compact set in $\mathbb{R}$ without isolated points.

Let $K \subset \mathbb{R}$ be a perfect set. The space of functions $f : K \to \mathbb{R}$ extendable to C^∞-functions on $\mathbb{R}$ equipped with the topology defined by the sequence of norms

$$\| f \|_q = |f|_q + sup \left\{ |(R_y^q f)^{(i)}(x)| \cdot |x-y|^{i-q}; x, y \in K, x \neq y, i = 0, 1, ...q \right\},$$

$q = 0, 1, \cdots$, where $|f|_q = sup\{|f^{(i)}(x)| : x \in K, i \leq q\}$ and $R^q_y f(x) = f(x) - T^q_y f(x)$ (the Taylor remainder) is called *the space of Whitney functions on* K and is denoted by $\mathcal{E}(K)$. By the Whitney Theorem, we can say that each function $f \in \mathcal{E}(K)$ is extendable to a $C^\infty-$function on the line. If there exists a linear continuous extension operator $L : \mathcal{E}(K) \to C^\infty(\mathbb{R})$, then we say that the compact set K has the *extension property.* (see e.g.[3])

In [3] the following result was proved for the regular generalized Cantor-type sets (see 2.5 for definition) and when $N_n = N$, $\forall n$:

Proposition 6 *If* $\liminf \alpha_n > N$, *then* $K^{(\alpha_n)}_N$ *does not have the extension property. If* $\limsup \alpha_n < N$, *then* $K^{(\alpha_n)}_N$ *has the extension property.*

And as a corollary it was given that:

Proposition 7 *For a compact set* $K^{(\alpha_n)}_N$, *let the limit* $\alpha = \lim \alpha_n$ *exist and be not equal to* N. *Then* $K^{(\alpha_n)}_N$ *has the extension property if and only if* $\lambda_0(K^{(\alpha_n)}_N) > 1$.

Also the logarithmic dimension of Cantor-type set K is related to the important linear topological invariant, namely the diametral dimension of the space $\mathcal{E}(K)$ (see e.g.[22]). Let X be a Fréchet space with fundamental system of neighborhoods (U_q), let $d_n(U_q, U_p)$ denote the $n-$th Kolmogorov diameter (see [19] for details) of U_q with respect to U_p. Then,

$$\Gamma(X) = \{(\gamma_n)^\infty_{n=0} : \forall p\ \exists q : \gamma_n \cdot d_n(U_q, U_p) \to 0 \text{ as } n \to \infty\}.$$

If two spaces X and Y are isomorphic, then $\Gamma(X) = \Gamma(Y)$.

We will consider the counting function corresponding to the diametral dimension

$$\beta(t) = \beta(U_p, U_q, t) = min\{dimL : t \cdot U_q \subset U_p + L\}, \quad t > 0.$$

The diametral dimension can be characterized in terms of β in the following way.

Corollary 1 ([3]) $(\gamma_n) \in \Gamma(X) \Leftrightarrow \forall p\, \exists q \,:\, \forall C\, \exists n_0 \,:\, \beta(U_p, U_q, C\gamma_n) \le n$ *for* $n \ge n_0$.

The main theorem about diametral dimension proved in [3] is the following:

Proposition 8 *Let* $X = \mathcal{E}(K)$ *with* $K = K^{(\alpha_n)}_{(N_n)}$, *let* p *and* q, $p < q$ *be fixed natural numbers. If* $t \le \frac{1}{5}\, l_n^{p-q}$, *then* $\beta(U_p, U_q, t) \le (q+1)\, N_1 \cdots N_n$. *If* $t \ge 5(q-p)!\, l_n^{p-q}$, *then* $\beta(U_p, U_q, t) \ge N_1 \cdots N_n$.

Using this theorem one can easily find the diametral dimension of $\mathcal{E}(K)$ for concrete compact set K. In particular for classical Cantor set we have $\beta(U_p, U_q, t) \sim t^{\frac{\log 2}{(q-p)\log 3}}$, which shows that the diametral dimension of $\mathcal{E}(K)$ is the same as $\Gamma(s)$, where $s = K(n^p)$ is the space of rapidly decreasing sequences. Here, $F \sim G$ means that for some C, t_0 we have

$$\frac{1}{C}\, F\left(\frac{t}{C}\right) \le G(t) \le C \cdot F(Ct), \quad t > t_0.$$

For the set $K^{(\alpha)}_N$ with the logarithmic dimension $\lambda_0 = \frac{\log N}{\log \alpha}$ we have from [3]:

Corollary 2 *Let* $X = \mathcal{E}(K^{(\alpha)}_N)$. *Then* $\beta(U_p, U_q, t) \sim \log^{\lambda_0} t$, $t \to \infty$.

Corollary 3 $\Gamma(\mathcal{E}(K^{(\alpha)}_N)) = \{(\gamma_n) \,:\, \exists M :\ \gamma_n \cdot \exp(-M\, n^{\frac{1}{\lambda_0}}) \to 0$ *as* $n \to \infty\}$.

Corollary 4 *If spaces of the type* $\mathcal{E}(K^{(\alpha)}_N)$ *are isomorphic, then the corresponding compact sets have the same logarithmic dimension.*

Corollary 5 *If* $\alpha < N$, *then the space* $\mathcal{E}(K^{(\alpha)}_N)$ *is isomorphic to a complemented subspace of* s, *but is not isomorphic to* s.

3.3 Local Interpolations

Given a compact set $K \subset \mathbb{R}$ and a sequence of distinct points $(x_n)_1^\infty \subset K$, let $e_n(x) = \prod_1^n (x - x_k)$ for $n \in \mathbb{N}_0 := \{0, 1, \cdots\}$, and $\prod_m^n(\cdots) = 1$ for $m > n$. Let $X(K)$ be a Fréchet space of continuous functions on K, containing all polynomials. By ξ_n we denote the linear functional $\xi_n(f) = [x_1, x_2, \cdots, x_{n+1}]\, f,\ f \in X(K),\ n \in \mathbb{N}_0$. ([12])

Let us give the definition and some properties of divided differences denoted by $[x_0, x_1, \cdots, x_j]f$.

The interpolating polynomial p_n, which assumes the same values as the function f at $x_0, x_1, \cdots, x_n$, was written by Isaac Newton (1642-1727) in the form

$$p_n = a_0\pi_0(x) + a_1\pi_1(x) + \cdots + a_n\pi_n(x) \tag{3.1}$$

where

$$\pi_0 = 1 \text{ and } \pi_i(x) = (x - x_0)(x - x_1)\cdots(x - x_{i-1}),\ 1 \le i \le n.$$

We may determine the coefficients a_j by setting

$$p_n(x_j) = f(x_j), \quad 0 \le j \le n.$$

We will write

$$a_j = [x_0, x_1, \cdots, x_j]f, \quad 0 \le j \le n$$

to emphasize its dependence on f and $x_0, x_1, \cdots, x_j$, and refer to a_j as the *$j-th$ divided difference.* Thus we may write (3.1) in the form

$$p_n = [x_0]f\,\pi_0(x) + [x_0, x_1]f\,\pi_1(x) + \cdots + [x_0, x_1\cdots x_n]f\,\pi_n(x)$$

which is Newton's divided difference formula for the interpolating polynomial. ([25])

It is that the divided difference $[x_0, x_1 \cdots x_n]f$ is a symmetric function of its arguments $x_0, x_1 \cdots x_n$.

Proposition 9 *(see e.g.[25]) The divided difference* $[x_0, x_1 \cdots x_n]f$ *can be expressed as the following symmetric sum of multiples of* $f(x_j)$,

$$[x_0, x_1 \cdots x_n]f = \sum_{r=0}^{n} \frac{f(x_r)}{\prod_{j\neq r}(x_r - x_j)},$$

where in the above product of n *factors,* r *remains fixed and* j *takes all values from* 0 *to* n*, excluding* r.

We can use this symmetric form to show that

$$[x_0, x_1, \cdots, x_n]\, f = \frac{[x_1, x_2, \cdots, x_n]\, f - [x_0, x_1, \cdots, x_{n-1}]\, f}{x_n - x_0}$$

It is also suitable to include the following property of divided differences from [19]:

Let x and the abscissas $x_0, x_1, \cdots, x_n$ be contained in an interval $[a, b]$ on which f and its first n derivatives are continuous, and let $f^{(n+1)}$ exist in the open interval (a, b). Then, there exists a number $\xi_x \in (a, b)$ such that

$$[x, x_0, x_1, \cdots, x_n]\, f = \frac{f^{(n+1)}(\xi_x)}{(n+1)!}.$$

Since this holds for any x belonging to an interval $[a, b]$ that contains all the abscissas x_j, we can replace n by $n-1$, put $x = x_n$, and obtain

$$[x_0, x_1, \cdots, x_n]\, f = \frac{f^{(n)}(\xi)}{n!},$$

where $\xi \in (x_0, x_n)$. Thus an n-th order divided difference, which involves $n+1$ parameters, behaves like a multiple of an n-th order derivative.

By these properties of divided differences we have;

Lemma 2 *If a sequence* $(x_n)_1^\infty$ *of distinct points is dense on a perfect compact set* $K \subset \mathbb{R}$*, then the system* $(e_n, \xi_n)_{n=0}^\infty$ *is biorthogonal and the sequence of functionals* $(\xi_n)_{n=0}^\infty$ *is total on* $X(K)$*, that is whenever* $\xi_n(f) = 0$ *for all* n*, it follows that* $f = 0$.

We will use the following convolution property of the coefficients of basis expansions from [12] as:

Lemma 3 *Let* $(a_k^{(s)})_k^\infty,\ s = 1, 2, 3,$ *be three sequences such that for a fixed superscript* s *all points in the sequence* $(a_k^{(s)})_k^\infty$ *are different. Let* $e_{ns} = \prod_{k=1}^{n}(x - a_k^{(s)})$ *and* $\xi_{ns}(f) = [a_1^{(s)}, a_2^{(s)} \cdots a_{n+1}^{(s)}]f$ *for* $n \in \mathbb{N}_0$. *Then*

$$\sum_{q=p}^{r} \xi_{p\,3}(e_{q\,2})\,\xi_{q\,2}(e_{r\,1}) = \xi_{p\,3}(e_{r\,1}), \quad for\ \ p \le r.$$

Using Lemma 3, we can construct biorthogonal functionals corresponding to the local interpolation of functions as in [12]. As an example, let us consider the case of generalized Cantor-type sets.

Suppose we have a chain of compact sets $K_0 \supset K_1 \supset \cdots \supset K_s \supset \cdots$ and finite systems of distinct points $(a_k^{(s)})_{k=1}^{M_s} \subset K_s$ for $s = 0, 1, \cdots$ such that some part of the knots on K_{s+1}, namely $(a_k^{(s+1)})_{k=1}^{T_{s+1}}$, belongs to the previous set $(a_k^{(s)})_{k=1}^{M_s}$. The sequences (T_s) and (M_s) can be specified later. Here we will take $N_s T_{s+1} = M_s \le M_{s+1}$.

We will define the function e_{ns} for any $s \ge 0$ and for $n = T_s + 1, \cdots, M_s$, as : $e_{ns} = \Pi_{k=1}^{n}(x - a_k^{(s)})$ for $x \in K_s$ and $e_{ns} = 0$ for $x \in K_0 \setminus K_s$. If $K_{s-1} \setminus K_s$ is closed for any $s \ge 1$, then it is clear that the functions e_{ns} are continuous on K_0. Let $\xi_{ns}(f) = [a_1^{(s)}, a_2^{(s)}, \cdots a_{n+1}^{(s)}]f$ with $a_{M_s+1}^{(s)} := a_{T_{s+1}+1}^{(s+1)}$. We can easily see that $\xi_{ns}(e_{m,\,s+1}) = 0$, because the number $\xi_{ns}(f)$ is defined by values of f at some points on $K_s \setminus K_{s+1}$ where $e_{m,\,s+1}$ is zero by definition and at some points from $(a_k^{(s+1)})_{k=1}^{T_{s+1}}$, which are zeros of the function $e_{m,\,s+1}$. It is clear that, $\xi_{n,\,s+1}(e_{ms}) = 0$ for $n > m$ because e_{ms} is a polynomial of degree m whereas $\xi_{n,\,s+1}(e_{ms})$ can be written as $n-$th derivative of e_{ms} by the properties of divided differences. On the other hand, for $n \le m$ the functional $\xi_{n,\,s+1}$ in general is not biorthogonal to e_{ms} as the $n-$th derivative of a polynomial of degree m with $n \le m$ may not be zero. For this reason we take the functional

$$\eta_{n,\,s+1} = \xi_{n,\,s+1} - \sum_{k=n}^{M_s} \xi_{n,\,s+1}(e_{ks})\,\xi_{ks},$$

which is biorthogonal, not only to all elements e_{ms}, but also, by the convolution property, to all e_{mj} with $j = 0, 1, \cdots, s-1$.

In [12] Goncharov gave the method of construction of bases in spaces of differentiable functions defined on fractal sets. His results are the following:

1. $\alpha_s \geq 2 \Rightarrow (e_M, \xi_M)$ *is a basis in* $\mathcal{E}(K_2^{(\alpha_n)})$.

2. $\forall\, K_2^{(\alpha_n)}$ $(e_{M,j,s}, \eta_{M,j,s})$ *is a basis provided a nondecreasing unbounded sequence* $(M_s)_{s=0}^{\infty}$ *of natural numbers of the form* $M_s = 2^{n_s}$ *is such that the sequence* $(2^{M_s} l_s^Q)_{s=0}^{\infty}$ *is bounded for some* Q.

We see that the first result corresponds to the case $\lambda_0 \leq 1$, where the second gives the basis for the spaces of Whitney functions given on any set of the type $K_2^{(\alpha_n)}$. Our aim is to generalize these results.

3.4 Polynomial basis in $\mathcal{E}(K(\Lambda))$ for Cantor-type sets with small logarithmic dimension

Theorem 2 *Assuming the existence of the limit, if* $\lambda_0(K^{(\alpha_n)}_{(N_n)}) < 1$, *then the sequence* $(e_M)_{M=0}^{\infty}$ *is a Schauder basis in the space* $\mathcal{E}(K^{(\alpha_n)}_{(N_n)})$.

Proof: We will follow the method suggested in [12]. By Lemma 3, the system $(e_M, \xi_M)_{M=0}^{\infty}$ is biorthogonal with a total sequence of functionals. Therefore, we will use the Dynin-Mityagin criterion defined above.

Without loss of generality we can assume $p = 2N_{n-u+1} \cdots N_{n-1}$. And given u we take $q + 1 = 2N_{n-v+1} \cdots N_{n-1}$ where $v = v(u)$ will be specified later. Let us fix $M = 2N_1 \cdots N_{n-1} + \nu$, where $0 \leq \nu = k_m\, N_1 \cdots N_m + k_{m-1}\, N_1 \cdots N_{m-1} + \cdots + k_1\, N_1 + k_0 < 2N1 \cdots N_{n-1}$ with $m \leq n-1\,, k_m \leq N_m\,, k_{m-1} \leq N_{m-1}\,, \ldots, k_1 \leq N_1$. According to the procedure we choose firstly $2N_1 \cdots N_{n-1}$ points of the type less than or equal to $n-1$ and separate the remaining ν points of $n-$th type into groups: $k_j\, N_1 \cdots N_j$ points (let us denote this set by X_j) are uniformly distributed on the basic intervals $I_{m,j}, m = 1, N_1, \cdots, N_1 \cdots N_j$. In this notation $e_M(x) = \Pi_{j=0}^m \Pi_{x_k \in X_j}(x - x_k)$.

For the first case let $M = 2N_1 \cdots N_{n-1}$ (i.e, $\nu = 0$ and the sets X_j are empty for $j \geq 1$). By the structure of the set $K(\Lambda)$, for $x \in K(\Lambda)$ we get

$\Pi_{x_k \in X_n} | x - x_k| \leq l_{n-1}^2 \, l_{n-2}^{2(N_{n-1}-1)} \, l_{n-3}^{2(N_{n-2}N_{n-1}-N_{n-1})} \cdots l_1^{2(N_2 \ldots N_{n-1} - N_3 \ldots N_{n-1})} \, l_0^{2(N_1 \ldots N_{n-1} - N_2 \ldots N_{n-1})}$.

Therefore,

$$| e_M |_0 \leq l_{n-1}^2 \, l_{n-2}^{2(N_{n-1}-1)} \cdots l_0^{2(N_1 \ldots N_{n-1} - N_2 \ldots N_{n-1})} = \Pi_1^M z_k, \tag{3.2}$$

where $(z_k)_1^M$ are arranged in nondecreasing order.

Now $e_M(x) = \prod_1^M (x - x_k) = (x - x_1)(x - x_2) \cdots (x - x_M)$. Hence

$$\begin{aligned} e_M^{(1)}(x) &= \prod_{k=2}^M (x - x_k) + (x - x_1) \left[\prod_{k=2}^M (x - x_k) \right]' \\ &= \prod_{k=2}^M (x - x_k) + (x - x_1) \prod_{k=3}^M (x - x_k) + (x - x_1)(x - x_2) \left[\prod_{k=3}^M (x - x_k) \right]' \\ &= \sum_{j=1}^M \prod_{k=1, k \neq j}^M (x - x_k). \end{aligned}$$

Similarly

$$e_M^{(2)}(x) = \sum_{l=1}^M \sum_{j=1}^M \prod_{k=1, k \neq j \neq l}^M (x - x_k).$$

Hence the $i-$th derivative of $e_M(x)$ represents the sum of $M!/(M-i)!$ products where every product contains $M - i$ terms of type $(x - x_k)$. So for $p < M$;

$$| e_M |_p \leq M!/(M-p)! \, \Pi_{p+1}^M z_k \leq M^p \, \Pi_{p+1}^M z_k.$$

In order to estimate $|| e_M ||_p$, fix $x, y \in K$, $i \leq p = 2N_{n-u+1} \cdots N_{n-1}$. Denote $(R_y^p e_N)^{(i)}(x)$ by R. Suppose at first, that x and y belong to the same basic interval

$I_{j,\,n-u+1}$. By the Lagrange form of the Taylor remainder, we find $\theta \in I_{j,\,n-u+1}$ such that

$$(R^p_y e_M)^{(i)}(x) = |e^{(p)}_M(\theta) - e^{(p)}_M(y)|\,\frac{(x-y)^{p-i}}{(p-i)!}$$

$$\Rightarrow |R|\cdot|\,x-y\,|^{\,i-p} = \frac{|e^{(p)}_M(\theta) - e^{(p)}_M(y)|}{(p-i)!} \leq |\,e^{(p)}_M(\theta) -\; e^{(p)}_M(y)\,|,$$

where $\theta \in I_{j,\,n-u+1}$. As above we get the bound $|\,e^{(p)}_M(\theta)| \leq M^p\,\Pi^M_{p+1} d_k(\theta)$ where $d_k(\theta) := |\,\theta - x_{i_k}|$ and it is a nondecreasing sequence. The interval $I_{j,\,n-u+1}$ contains λ points (with $p/N_{n-u+1} \leq \lambda \leq p$) of the set $(x_k)^M_1$. But $d_k(\theta) \leq z_k$ for $k > \lambda$. Therefore, $|R|\cdot|\,x-y\,|^{\,i-p} \leq |\,e^{(p)}_M(\theta) -\; e^{(p)}_M(y)\,| \leq 2M^p\,\Pi^M_{p+1} z_k$.

Suppose now that $|\,x-y\,| \;\geq\; h_{n-u} \;=\; (l_{n-u-1} - N_{n-u}l_{n-u})/N_{n-u} - 1 \;\geq\; \frac{1}{2N_{n-u}-1}\,l_{n-u}$. Then for any j with $i \leq j \leq p$, $|\,x-y\,|^{\,j-p} \leq (2N_{n-u}-1)^{p-j}\,l^{j-p}_{n-u}$. Hence we get the bound

$$\begin{aligned}
|\,e^{(j)}_M(y)\,|\cdot|\,x-y\,|^{\,j-p} &\leq M^j\,\Pi^M_{j+1} z_k\,(2N_{n-u}-1)^{p-j}\,l^{j-p}_{n-u}\\
&\leq M^p\,(2N_{n-u}-1)^{p-j}\,\Pi^M_{p+1} z_k\,(z_{j+1}\,z_{j+2}\cdots z_p)\,l^{j-p}_{n-u}\\
&= M^p\,(2N_{n-u}-1)^{p-j}\,\Pi^M_{p+1} z_k\,(z_{j+1}\,z_{j+2}\cdots z_p)\,\left[l^{p-j}_{n-u}\right]^{-1}\\
&\leq M^p\,(2N_{n-u}-1)^{p}\,\Pi^M_{p+1} z_k\\
&\leq M^p\,(2N_{n-u})^{p}\,\Pi^M_{p+1} z_k,
\end{aligned}$$

as $z_{j+1}, \cdots, z_p \leq l_{n-u}$. Hence,

$$\begin{aligned}
|R|\cdot|\,x-y\,|^{\,i-p} &= |e^{(i)}_M(x)|\cdot|x-y|^{\,i-p} - |e^{(i)}_M(y)|\cdot|x-y|^{\,i-p} - \cdots - |e^{(p)}_M(y)|\cdot\frac{|x-y|^{\,p-p}}{(p-i)!}\\
&\leq |\,e^{(i)}_N(x)\,|\cdot|\,x-y\,|^{\,i-p} + \sum_{j=i}^{p}|\,e^{(j)}_N(y)\,|\cdot|\,x-y\,|^{\,j-p}/(j-i)!\\
&\leq (2N_{n-u}M)^p\,\Pi^M_{p+1} z_k\left\{1 + \sum_{j=i}^{p} 1/(j-i)!\right\}\\
&\leq (e+1)\,(2N_{n-u}M)^p\,\Pi^M_{p+1} z_k\\
&\leq 4\,(2Nn - uM)^p\,\Pi^M_{p+1} z_k.
\end{aligned}$$

Thus,

$$\begin{aligned}||e_M||_p &= |e_M|_p + sup\left\{|(R^p_y f)^{(i)}(x)| \cdot |x-y|^{i-p}; x,y \in K(\Lambda), x \neq y, i=0,1,...q\right\} \\ &\leq M^p\, \Pi^M_{p+1} z_k + 4\,(2N_{n-u}M)^p\, \Pi^M_{p+1} z_k \\ &\leq 5\,(2N_{n-u})^p\, M^p\, \Pi^M_{p+1} z_k.\end{aligned}$$

To estimate the dual $q-$th norm of ξ_M we suppose that M is large enough, enumerate the first $M+1$ points of the sequence $(x_n)_1^\infty$ in increasing order and use the bound (1) from [15]:

$$|\,[x_1, \cdots, x_{M+1}]f\,| \leq 2^{M-q}\,|\,\tilde{f}\,|_q^{([0,1])}(min\, \Pi_{m=1}^{M-q}|\, x_{a(m)} - x_{b(m)}|\,)^{-1}, \tag{3.3}$$

where $\tilde{f} \in C^\infty[0,1]$ is any extension of f on $[0,1]$; min is taken over all $1 \leq j \leq M+1-q$ and all possible chains of strict embeddings $[x_{a(0)}, \cdots, x_{b(0)}] \subset [x_{a(1)}, \cdots, x_{b(1)}] \subset \cdots \subset [x_{a(M-q)}, \cdots, x_{b(M-q)}]$ with $a(0)=j$, $b(0)=j+q, \cdots, a(M-q)=1$, $b(M-q)=M+1$. Here, given $a(k)$, $b(k)$ we take $a(k+1)=a(k)$, $b(k+1)=b(k)+1$ or $a(k+1)=a(k)-1$, $b(k+1)=b(k)$. We will denote by Π the minimizing product above.

Let us consider all possible locations of $q+1 = 2N_{n-v+1}\cdots N_{n-1}$ consecutive points $(x_{j+k})_{k=0}^q$ from $(x_n)_1^{M+1}$. Every interval of the length l_{n-v} contains $2N_{n-v+1}\cdots N_{n-1}$ such points, which is equal to $q+1$. Therefore the product above can take its minimal value only if all $q+1$ points are situated on the same interval of this length. Fix this interval $I_{i,n-v}$. Let it contain μ points from $(x_n)_1^{M+1}$. Each of N_{n-v+1} subintervals

$$I_{N_{n-v+1}i-(N_{n-v+1}-1),n-v+1}, I_{N_{n-v+1}i-(N_{n-v+1}-2),n-v+1}, \cdots, I_{N_{n-v+1}i,n-v+1}$$

of $I_{i,n-v}$ contains exactly $2N_{n-v+2}\cdots N_{n-1}$ points, therefore the first $\mu-q-1$ terms of the product Π are larger than the length of the gap h_{n-v}. The estimation of terms of Π is as follows: If we choose the first subinterval, $[0, l_{n-v+1}]$, to be the interval where all $q+1$ points are located, then we will have

$$\Pi \geq {h_{n-v}}^{2N_{n-v+2}\cdots N_{n-1}}\,(2h_{n-v})^{2N_{n-v+2}\cdots N_{n-1}} \cdots ((N_{n-v+1}-1)h_{n-v})^{2N_{n-v+2}\cdots N_{n-1}}$$

in the interval of length l_{n-v} and hence for the whole interval,

$$\begin{aligned}\Pi \;\geq\; & (N_{n-v+1}-1)!^{2N_{n-v+2}\cdots N_{n-1}} \cdots (N_1-1)!^{2N_2\cdots N_{n-1}} \\ & h_{n-v}{}^{(N_{n-v+1}-1)(2N_{n-v+2}\cdots N_{n-1})} \cdots h_0{}^{(N_1-1)(2N_2\cdots N_{n-1})}.\end{aligned}$$

If we choose the middle subinterval (one of the middle ones if N_{n-v+1} is odd), then we will have

$$\begin{aligned}\Pi \;\geq\; & \left[(N_{n-v+1}-1/2)!^{2N_{n-v+2}\cdots N_{n-1}} \cdots (N_1-1/2)!^{2N_2\cdots N_{n-1}}\right]^2 \\ & h_{n-v}{}^{(N_{n-v+1}-1)(2N_{n-v+2}\cdots N_{n-1})} \cdots h_0{}^{(N_1-1)(2N_2\cdots N_{n-1})}.\end{aligned}$$

As we see they differ just by a constant and moreover these inequalities show that the remaining terms of Π can be estimated from below by the lengths of the gaps $h_{n-v}, h_{n-v-1}, \cdots, h_0$. Hence we get the product as in (3.2) with a constant factor, say K, but l_k should be replaced by h_k and the smallest q terms are absent. We have

$$h_k/l_k \sim \frac{1}{N_n} > \frac{1}{N}$$

as $N_n < N$.

Therefore, after removing q points we have

$$\begin{aligned}\Pi &= (min\, \Pi_{m=1}^{M-q} |\, x_{a(m)} - x_{b(m)}|\,)^{-1} \\ &\geq K \left(\frac{1}{N}\right)^{M-q} \Pi_{q+1}^{M} z_k.\end{aligned}$$

Now consider the general case and let $M = 2N_1 \cdots N_{n-1} + \nu$, where $0 \leq \nu = k_m\, N_1 \cdots N_m + k_{m-1}\, N_1 \cdots N_{m-1} + \cdots + k_1\, N_1 + k_0 < 2N1 \cdots N_{n-1}$ with $m \leq n-1\,, k_m \leq N_m\,, k_{m-1} \leq N_{m-1}\,, \ldots, k_1 \leq N_1$. Now every interval of length l_{r_j} contains k_{r_j} points from the set X_{r_j}. So, in this case we have

$$\begin{aligned}\Pi_{x_k \in X_{r_j}} |\, x - x_k| \;\leq\; & l_{r_j-1}^{k_{r_j}}\, l_{r_j-2}^{k_{r_j}(N_{n-1}-1)}\, l_{r_j-3}^{k_{r_j}(N_{n-2}N_{n-1}-N_{n-1})} \cdots \\ & \cdots l_1^{k_{r_j}(N_2\cdots N_{n-1}-N_3\cdots N_{n-1})}\, l_0^{k_{r_j}(N_1\cdots N_{n-1}-N_2\cdots N_{n-1})}.\end{aligned}$$

So,

$$|\,e_M|_0 \leq \Pi_{j=0}^{m}\, l_{r_j-1}^{k_{r_j}}\, l_{r_j-2}^{k_{r_j}(N_{n-1}-1)} \cdots l_0^{k_{r_j}(N_1\cdots N_{n-1}-N_2\cdots N_{n-1})} = \Pi_1^M z_k, \qquad (3.4)$$

where $(z_k)_1^M$ are arranged in nondecreasing order. Everything we have for the previous case will be the same for this case up to the estimation of the minimizing product. For this estimation again consider all possible locations of $q+1$ consecutive points $(x_{j+k})_{k=0}^q$ from $(x_n)_1^{M+1}$. This time every interval of the length l_{n-v} contains more than $2N_{n-v+1}\cdots N_{n-1}$ such points (in fact it contains μ points where $2N_{n-v+1}\cdots N_{n-1} \leq \mu \leq 2N_{n-v}\cdots N_{n-1}$). Therefore, again the product above can take its minimal value if all $q+1$ points are situated on the same interval of this length. Now each of N_{n-v+1} subintervals $I_{N_{n-v+1}i-(N_{n-v+1}-1),\,n-v+1}, I_{N_{n-v+1}i-(N_{n-v+1}-2),\,n-v+1}, \cdots, I_{N_{n-v+1}i,\,n-v+1}$ of $I_{i,\,n-v}$ contains at most $2N_{n-v+1}\cdots N_{n-1}$ points (in fact each contains η points where $2N_{n-v+2}\cdots N_{n-1} \leq \eta \leq 2N_{n-v+1}\cdots N_{n-1}$) , therefore again the first $\mu-q-1$ terms of the product Π are larger than the length of the gap h_{n-v}. For this case the estimation of Π is as follows: If we choose the first subinterval, $[0, l_{n-v+1}]$, to be the interval where all $q+1$ points are located, then we will have for l_{n-v}

$$\Pi \geq {h_{n-v}}^{2N_{n-v+2}\cdots N_{n-1}}\,(2h_{n-v})^{2N_{n-v+2}\cdots N_{n-1}} \cdots ((N_{n-v+1}-1)h_{n-v})^{2N_{n-v+2}\cdots N_{n-1}}$$

and hence for the whole interval

$$\begin{aligned}\Pi \;\;\geq\;\; & (N_{n-v+1}-1)!^{2N_{n-v+2}\cdots N_{n-1}} \cdots (N_1-1)!^{2N_2\cdots N_{n-1}} \\ & {h_{n-v}}^{(N_{n-v+1}-1)(2N_{n-v+2}\cdots N_{n-1})} \cdots {h_0}^{(N_1-1)(2N_2\cdots N_{n-1})}.\end{aligned}$$

If we choose the middle subinterval (one of the middle ones if N_{n-v+1} is odd), then we will have

$$\begin{aligned}\Pi \;\;\geq\;\; & [(N_{n-v+1}-1/2)!^{2N_{n-v+2}\cdots N_{n-1}} \cdots (N_1-1/2)!^{2N_2\cdots N_{n-1}}]^2 \\ & {h_{n-v}}^{(N_{n-v+1}-1)(2N_{n-v+2}\cdots N_{n-1})} \cdots {h_0}^{(N_1-1)(2N_2\cdots N_{n-1})}\end{aligned}$$

which is exactly the same estimation for the first case. That's why, after removing q points we will have the same inequality as

$$\begin{aligned}\Pi \;\;=\;\; & (min\,\Pi_{m=1}^{M-q}|\,x_{a(m)}-x_{b(m)}|\,)^{-1} \\ \geq\;\; & K\left(\frac{1}{N}\right)^{M-q}\,\Pi_{q+1}^{M} z_k.\end{aligned}$$

In addition, by the open mapping theorem for a given q there exists $r \in \mathbb{N}$, $C_q > 0$ such that

$$inf|\,\tilde{f}\,|_q^{([0,1])} \le C_q\,||\,f\,||_r \tag{3.5}$$

for any $f \in \mathcal{E}(K(\Lambda))$. Here inf is taken over all possible extensions of f to $\tilde{f}$ on $[0,1]$. As $|\,\xi|_{-q} = \sup\{|\,\xi(f)|\,:\,||\,f\,|| \le 1\}$, we have

$$\|\,e_M\|_p \cdot |\,\xi|_{-q} \le \frac{1}{K}\,C_q\,5\,(2N_{n-u})^p\,M^p\;\;\Pi_{p+1}^M\,z_k\,2^{M-q}\,\frac{1}{\left(\frac{1}{N}\right)^{M-q}\Pi_{q+1}^M\,z_k}.$$

As $N_n \le N$,for all n, we have

$$\begin{aligned}\|\,e_M\|_p \cdot |\,\xi|_{-q} &\le \frac{1}{K}\,C_q\,5\,(2N)^p\,M^p\;\;2^{M-q}\,N^{M-q}\Pi_{p+1}^q\,z_k\\ &\le \frac{1}{K}\,C_q\,5\,(2N)^p\,M^p\;\;2^{M}\,N^{M}\Pi_{p+1}^q\,z_k\end{aligned}$$

For the estimation of the product $\Pi_{p+1}^q\,z_k$ let us take into account only the terms z_k corresponding to the points from the set X_{r_0} because if we include the points from other sets X_{r_j} , this will only decrease product. Thus, removing terms after $q-$th one we get

$$l_{n-1}^2\,l_{n-2}^{2(N_{n-1}-1)}\,l_{n-3}^{2(N_{n-2}N_{n-1}-N_{n-1})}\,\cdots l_{n-v+1}^{2(N_{n-v+2}\cdots N_{n-1}-N_{n-v+3}\cdots N_{n-1})}.$$

Now we have to remove p smallest terms of this product. After removing we get,

$$\Pi_{p+1}^q\,z_k\;\le l_{n-u}^{2(N_{n-v+1}\cdots N_{n-1}-N_{n-v+2}\cdots N_{n-1})}\,\cdots l_{n-v+1}^{2(N_{n-v+2}\cdots N_{n-1}-N_{n-v+3}\cdots N_{n-1})} = l_1^\kappa$$

with

$$\begin{aligned}\kappa &= 2(N_{n-v+1}\cdots N_{n-1} - N_{n-v+2}\cdots N_{n-1})\alpha_1\cdots\alpha_{n-u-1}+\cdots\\ &\qquad\qquad \cdots + 2(N_{n-v+2}\cdots N_{n-1} - N_{n-v+3}\cdots N_{n-1})\alpha_1\cdots\alpha_{n-v}\\ &= 2(N_{n-u+2}\cdots N_{n-1})(N_{n-u+1}-1)\alpha_1\cdots\alpha_{n-u-1}+\cdots\\ &\qquad\qquad \cdots + 2(N_{n-v+3}\cdots N_{n-1})(N_{n-u+2}-1)\alpha_1\cdots\alpha_{n-v}\\ &\ge [2(N_{n-u+2}\cdots N_{n-1})(N_{n-u+1}-1)+\cdots+2(N_{n-v+3}\cdots N_{n-1})(N_{n-u+2}-1)]\alpha_1\cdots\alpha_{n-u-1}\\ &\ge (v-u-1)2(N_{n-u+2}\cdots N_{n-1})(N_{n-u+1}-1)(\alpha_1\cdots\alpha_{n-u-1})\\ &\ge (v-u-1)2(N_{n-u+2}\cdots N_{n-1})(N_{n-u+1}-1)(N_1\cdots N_{n-u-1})^{\frac{1}{1-\epsilon}}\end{aligned}$$

because

$$\lambda_0(K) = \liminf_n \frac{\log N_1\cdots N_n}{\log\alpha_1\cdots\alpha_n} < 1$$

implies

$$\frac{\log N_1 \cdots N_n}{\log \alpha_1 \cdots \alpha_n} < 1 - \epsilon \,, \forall n \geq N \,, \text{for some}\, N \in \mathbb{N}$$

and hence

$$\frac{\log N_1 \cdots N_{n-u-1}}{\log \alpha_1 \cdots \alpha_{n-u-1}} < 1 - \epsilon \,, \forall n \geq N \,, \text{for some}\, N \in \mathbb{N}.$$

Taking into account the bound $M < 2N_1 \cdots N_n$, we obtain

$$\begin{aligned} \| e_M \|_p \cdot |\, \xi|_{-q} &\leq \frac{1}{K} C_q \, 5 \, (2N)^p \, M^p \;\; 2^M \, N^M \Pi_{p+1}^q \, z_k \\ &\leq \frac{1}{K} C_q \, 5 \, (2N)^p \, (2N_1 \cdots N_n)^p \;\; (2N)^{2N_1 \cdots N_n} \, l_1^\kappa. \end{aligned}$$

Now let $(2N)^p \, (2N_1 \cdots N_n)^p \;\; (2N)^{2N_1 \cdots N_n} \, l_1^\kappa = B$. Then,

$$\begin{aligned} \log B &= p\log(2N) + p\log(2N_1 \cdots Nn) + (2N_1 \cdots Nn)\log(2N) + \kappa\log(l_1) \\ &= -(\kappa\log(l_1) - p\log(2N) - p\log(2N_1 \cdots Nn) - (2N_1 \cdots Nn)\log(2N)) \end{aligned}$$

We want the term inside the paranthesis to go to infinity when $n \to \infty$. So,

$$\kappa\log(l_1) - (2N_1 \cdots Nn)\log(2N)$$

should go to ∞, which is same as

$$\begin{aligned} & (v-u-1)\, 2(N_{n-u+2} \cdots N_{n-1})\, (N_{n-u+1} - 1)(N_1 \cdots N_{n-u-1})^{1+\epsilon} \, \log(l_1) - \\ & \qquad - \; 2(N_1 \cdots N_n) \log(2N) \to \infty \\ \iff & (v-u-1)\, \frac{2(N_1 \cdots N_n)}{N_{n-u} N_{n-u+1} N_n}\, (N_{n-u+1} - 1)(N_1 \cdots N_{n-u-1})^{\epsilon} \, \log(l_1) - \\ & \qquad - \; 2(N_1 \cdots N_n) \log(2N) \to \infty \\ \iff & 2(N_1 \cdots N_n) \left[(v-u-1)\, \frac{(N_{n-u+1} - 1)}{N_{n-u} N_{n-u+1} N_n}\, (N_1 \cdots N_{n-u-1})^{\epsilon} \, \log(l_1) - \; \log(2N) \right] \to \infty \\ \iff & (v-u-1)\, \frac{(N_{n-u+1} - 1)}{N_{n-u} N_{n-u+1} N_n}\, (N_1 \cdots N_{n-u-1})^{\epsilon} \, \log(\frac{1}{l_1}) - \; \log(2N) > 0 \\ \iff & (v-u-1)\, \frac{(N_{n-u+1} - 1)}{N_{n-u} N_{n-u+1} N_n}\, (N_1 \cdots N_{n-u-1})^{\epsilon} \, \log(\frac{1}{l_1}) - \; \log(2N) > 0 \end{aligned}$$

As a result,the value v such that

$$(v-u-1)\, (N_{n-u+1} - 1)(N_1 \cdots N_{n-u-1})^{\epsilon} \, \log(\frac{1}{l_1}) \; > \; N_{n-u} N_{n-u+1} N_n \log(2N)$$

gives the desired conclusion. □

3.5 Existence of basis in the general case

We will use the modified definition of biorthogonal functionals as introduced in [12]. Given a nondecreasing sequence of natural numbers $(n_s)_0^\infty$, let $M_s = 2N_{s+1}\dots N_{s+n_s}$, $T_s^{(l)} = \frac{1}{N_s}M_{s-1}+1$, $T_s^{(r)} = \frac{1}{N_s}M_{s-1}$ for $s \geq 1$ and $T_0 = 0$. Here, (l) and (r) mean *left* and *remaining* respectively. For the fixed basic interval $I_{j,s} = [a_{j,s}, b_{j,s}]$ we choose the sequence of points $(x_{n,j,\,s})_{n=1}^\infty$ using the same procedure as before.

Let

$$e_{M,1,0}(x) = \prod_{n=1}^{M}(x - x_{n,1,0}) = \prod_{1}^{M}(x - x_n)$$

for $x \in K(\Lambda)$, $M = 0, 1, \cdots, M_0$. For $s \geq 1$, $j \leq 2N_s \dots N_{n-1}$ let $e_{M,j,s} = \prod_{n=1}^{M}(x - x_{n,j,\,s})$ if $x \in K(\Lambda)\cap I_{j,s}$ and $e_{M,j,s} = 0$ on $K(\Lambda)$ otherwise. Here, $M = T_s^{(a)}, T_s^{(a)}+1, \cdots, M_s$ with $a = l$ for $j = N_s c, c \in \mathbb{N}$ and $a = r$ if j is not a multiple of N_s.

Biorthogonal functionals are given in the following way: for $s = 0, 1, \cdots;\ j = 1, 2, \cdots, 2N_s \dots N_{n-1}$, and $M = 0, 1, \cdots$, let $\xi_{M,j,\,s}(f) = [x_{1,j,\,s}, \cdots, x_{M+1,j,\,s}]f$. Set $\eta_{M,1,0} = \xi_{M,1,0}$ for $M \leq M_0$. Every basic interval $I_{j,s}$, $s \geq 1$, is a subinterval of a certain $I_{i,s-1}$ with $j = N_s i - (N_s - 1), N_s i - (N_s - 2), \cdots, N_s i$. Let

$$\eta_{M,j,\,s}(f) = \xi_{M,j,\,s}(f) - \sum_{k=M}^{M_{s-1}} \xi_{M,j,\,s}(e_{k,\,i,\,s-1})\,\xi_{k,\,i,\,s-1}(f)$$

for $M = T_s^{(a)}, T_s^{(a)}+1, \cdots, M_s$. Clearly, for $M > M_{s-1}$ the subtracted sum above is absent. Thus on the interval $I_{i,s-1}$ we consider polynomials $e_{M,i,\,s-1}$ up to the degree M_{s-1}. The functional $\xi_{M_{s-1},\,i,\,s-1}$ is defined by $M_{s-1}+1$ points, $\frac{1}{N_s}M_{s-1}+1$ of them belong to the left subinterval $I_{N_s i-(N_s-1),s}$. They are just the zeros of the first polynomial on this subinterval. The other $\frac{N_s-1}{N_s}M_{s-1}$ points give the zeros of the remainig $(N_s - 1)$ subintervals, $e_{T_s^{(r)},\,N_s i-(N_s-2)i,\,s}, e_{T_s^{(r)},\,N_s i-(N_s-3)i,\,s} \cdots, e_{T_s^{(r)},\,Ni,\,s}$. By the arguments in Section 1, we see that the system $(e, \eta) := (e_{M,\,j,\,s}, \eta_{M,\,j,\,s})_{s=0,\,j=1,\,M=T_s}^{\infty,\ \ 2N_s\dots N_{n-1},\ \ M_s}$ is biorthogonal with the total on the $\mathcal{E}(K(\Lambda))$ sequence of functionals. It satisfies the condition of the Dynin-Mityagin criterion, if the choice of the sequence $(n_s)_0^\infty$ is suitable.

Theorem 3 *$\forall K^{(\alpha_n)}_{(N_n)}$, if a nondecreasing unbounded sequence $(M_s)_{s=0}^{\infty}$ of natural numbers of the form $M_s = 2N_{s+1}\dots N_{s+n_s}$ is such that the sequence $(2N_s^{M_s} l_s^Q)_{s=0}^{\infty}$ is bounded for some Q, then $(e_{M,j,s}, \eta)$ is a basis in the space $\mathcal{E}(K^{(\alpha_n)}_{(N_n)})$.*

Proof: Again we will follow the same method suggested in [12]. We can assume that for some Q and $s \geq 1$,

$$2N_s^{M_s} l_s^Q \leq 1$$

where $(M_s)_{s=0}^{\infty}$ is a non-decreasing unbounded sequence of natural numbers of the form $M_s = 2N_{s+1}\dots N_{s+n_s}$.

Let us take $p = 2N_{n-u+1}\dots N_{n-1}$ and q of the form $2N_{n-v}\dots N_{n-1}$ such that $q \geq p+6Q+1$. Fix s with $q < \frac{2N_{s+1}\dots N_{s+n_{s-1}-1}}{N_s}$ and $j \leq 2N_s\dots N_{n-1}$. Fix $\frac{1}{N}M_{s-1} \leq \frac{1}{N_s}M_{s-1} \leq M \leq M_s$. Let $M = 2N_{s+1}\dots N_{s+n} + \nu$ with $n_{s-1} - 1 \leq n \leq n_s$ and $0 < \nu < 2N_{s+1}\dots N_{s+n}$. Then the function $e_{M,j,s}$ has zeros at all endpoints of the the type less than or equal to $s+n-1$ on $I_{j,s}$ and some end points of the type $s+n$. After similar calculations as in proof of Theorem 2 we can show that

$$\|e_{M,j,s}\|_p \leq 5(2N_{n-u})^p M^p \Pi_{p+1}^M z_k.$$

Here the nondecreasing set $(z_k)_1^M$ consists of the lengths $l_{s+n}, l_{s+n-1}, \cdots, l_s$ taken from the product

$$l_{s+n}^2 l_{s+n-1}^{2(N_{n-1}-1)} l_{s+n-2}^{2(N_{n-2}N_{n-1}-N_{n-1})} \cdots l_{s+n}^{2(N_1\dots N_{n-1}-N_2\dots N_{n-1})}$$

corresponding to the set X_{s+n}. Note that the points from $K(\Lambda) \setminus I_{j,s}$ have no influence on the estimation of $\|e_{M,j,s}\|_p$ for $p < M$, since $dist(I_{j,s}, K(\Lambda) \setminus I_{j,s}) = h_{s-1}$ is larger than l_s.

Now we have to estimate the functional $\eta_{M,j,s}$. Without loss of generality let $j = N_1 i$. The interval $I_{N_1 i,s}$ is a subinterval of $I_{i,s-1}$. Therefore,

$$\eta_{M,N_1 i,s} = \xi_{M,N_1 i,s} - \sum_{k=M}^{M_{s-1}} \xi_{M,N_1 i,s}(e_{k,i,s-1})\xi_{k,i,s-1} \tag{3.6}$$

Repeating (3.3) and (3.5) we get,

$$|\xi_{M,N_1 i,s}(f)| \leq C_q \, 2^{M_s} \, \|f\|_r {\Pi_M}^{-1} \tag{3.7}$$

where Π_M denotes the minimal product corresponding to the functional $\xi_{M,N_1 i,s}$. This product contains $M-q$ terms of type $|x_{n,N_1 i,s} - x_{m,N_1 i,s}|$.

As in the proof of Theorem 2, due to the fact that $l_s < N h_s$, we get

$$ {\Pi_M}^{-1} \leq \left(\frac{l_s}{h_s}\right)^{M-q} \left(\Pi_{q+1}^M z_k\right)^{-1} \leq N^M \left(\Pi_{q+1}^M z_k\right)^{-1} \leq N^{M_s} \left(\Pi_{q+1}^M z_k\right)^{-1} \tag{3.8}$$

Now, we will try to estimate the norm $|\cdot|_{-q}$ of the subtracted sum (and consequently of $\eta_{M,N_1 i,s}$) above by the expression similar to the right part of (3.8). Now, $\frac{1}{N} M_{s-1} \leq \frac{1}{N_s} M_{s-1} \leq M \leq M_{s-1}$. First note that, for any k, $M \leq k \leq M_{s-1}$, we have

$$\begin{aligned} |\ \xi_{M,N_1 i,s}(e_{k,i,s-1})| &= |{e_{k,i,s-1}}^m(\theta)| = \frac{k!}{M!(k-M)!} \prod_{M+1}^{k} (x - x_{n,i,s-1}) \\ &\leq \binom{k}{M} {l_{s-1}}^{k-M} \leq 2^k \, l_{s-1}^{k-M} \end{aligned}$$

If $M = M_{s-1}$, then $\eta_{M,N_1 i,s} = \xi_{M,N_1 i,s} - \xi_{M,i,s-1}$ and as

$$|\ \xi_{M,i,s-1}|_{-q} \leq C_q \, 2^{M_s} \, N^{M_s} \, {\Pi_M}^{-1}$$

we have the bound for η. Now, to look at the remaining cases, assume that $M + 1 = \frac{1}{N_{s-1}} M_{s-1} + \nu$ with $1 \leq \frac{1}{N_{s-1}} M_{s-1}$. from $M+1$ points on $I_{N_1 i,s}$ that define the functional $\xi_{M,N_1 i,s}$, we have $2N_1 \ldots N_{n_{s-1}-1}$ endpoints of type less then or equal to $s + n_{s-1} - 1$ and ν endpoints of the type $s + n_{s-1}$.

Fix k such that $M \leq k \leq M_{s-1}$. Denote by Z_k the set $(x_{n,i,s-1})_{n=1}^{k+1}$, which defines the functional $\xi_{k,i,s-1}$.

As in (3.7), we have the bound

$$|\ \xi_{k,i,s-1}\ |_{-q} \leq C_q \, 2^k \, {\Pi_k}^{-1} \tag{3.9}$$

where Π_k denotes the minimal product $\Pi_{t=1}^{k-q} y_t$ corresponding to the functional $\xi_{k,i,s-1}$. The terms of Π_k are arranged in increasing order.

The interval $I_{i,s-1}$ contains $2N_1 \dots N_{n_{s-1}}$endpoints of type $\leq\ s+n_{s-1}-1$. Therefore, the chosen $k+1$ points occupy all endpoints of the type $\leq\ s+n_{s-1}-2$ and some endpoints of the type $\leq\ s+n_{s-1}-1$. If $k = M_{s-1}$, hen we get one endpoint of type $s+n_{s-1}$. In general we can write $k = N_s \cdot m + \mu$, where $\mu < N_s$.

For the first case suppose $\mu = 0$ and $k = N_s \cdot m$. Then the interval $I_{N_1 i-(N_1-1),s}$ contains $m+1$ points of Z_k which has $k+1$ points, whereas the remaining subintervals $I_{N_1 i-(N_1-2),s}, \cdots, I_{N_1 i,s}$ contains only m.

We have $\frac{1}{N_s} M_{s-1} \leq M \leq k \leq M_{s-1} = 2N_s \dots N_{s+n_s-1}$ and $m = \frac{k}{N_s}$. So,

$$m \geq \frac{M_{s-1}}{N_{n_s} N_{n_s}} = \frac{2N_{s+1} \dots N_{s+n_{s-1}-1}}{N_s}$$

But we have

$$q < \frac{2N_{s+1} \dots N_{s+n_{s-1}-1}}{N_s}$$

hence $m \geq q$ and we can choose $q+1$ consecutive points from Z_k on $Z_k \cap I_{N_1 i-(N_1-1),s}$ in order to get minimum value for Π_k.

Consider the decomposition

$$\Pi_k = \Pi_{t=1}^{m+1-q} y_t \cdot \Pi_{t=m+2-q}^{M-q} y_t \cdot \Pi_{t=M+1-q}^{k-q} y_t = \pi_1 \cdot \pi_2 \cdot \pi_3$$

We know $m \leq M$, so the density of points is greater in $I_{N_1 i,s}$, which contains $M+1$ points, than $I_{N_1 i-(N_1-1),s}$, which contains $m+1$ points. So, the value of π_1 is greater than or equal to the product of first $m+1-q$ terms of Π_M. In fact, π_1 is equal to this product only in case when $M = \frac{1}{N_s} M_{s-1}$ and $k = M_{s-1}$. Then the distribution of points of $Z_k \cap I_{N_1 i-(N_1-1),s}$ completely repeats the distribution of $Z_M := (x_{n,N_1 i,s})_{n=1}^{M+1}$. In all other cases we have $m < M$ and the density of distribution of points from $Z_k \cap I_{N_1 i-(N_1-1),s}$ is smaller than the one for Z_M.

On the other hand, any term of π_2 is not smaller than $l_s + h_{s-1}$, so it is larger than any term of Π_M. Hence, $\pi_1 \cdot \pi_2 > \Pi_M$. Any term of π_3 is larger than h_{s-1} and hence,

$$\Pi_k \geq \Pi_M \cdot h_{s-1}^{k-M} \tag{3.10}$$

Now, let $k = N_s \cdot m + \mu$, where $\mu \neq 0$. In this case, $\mu + 1$ subintervals include $m + 1$ points and the rest will include m points. Now, we have,

$$\begin{aligned} m &= \frac{k}{N_s} - \frac{\mu}{N_s} \geq \frac{M}{N_s} - \frac{\mu}{N_s} \geq \frac{1}{N_s}\left(\frac{1}{N_s} M_{s-1}\right) - \frac{\mu}{N_s} \\ &= \frac{2N_1 \dots N_{n_{s-1}}}{{N_s}^2} - \frac{\mu}{N_s} > \frac{2N_1 \dots N_{n_{s-1}}}{{N_s}^2} - 1 \end{aligned}$$

which means $m + 1 > q$ and hence $m + 1 \geq q + 1$. So, also in this case, in order to get the minimal value for Π_k we need to choose one of the intervals with $m+1$ points. As $m = \frac{k}{N_s} - \frac{\mu}{N_s} \leq \frac{M_{s-1}}{N_s} - \frac{\mu}{N_s} \leq M - \frac{\mu}{N} < M$ we can argue as before for the estimation of the product Π_k.

Taking into account (3.9), (3.10) and the bound for $|\,\xi_{M,N_1 i,s}(e_{k,i,s-1})|$, we see that

$$\begin{aligned} |\,\xi_{M,N_1 i,s}(e_{k,i,s-1})| \cdot |\,\xi_{k,i,s-1}\,|_{-q} &\leq C_q\, 2^k\, {\Pi_k}^{-1}\, 2^k\, l_{s-1}^{k-M} \\ &\leq C_q\, 2^{2k}\, (l_{s-1}/h_{s-1})^{k-M}\, {\Pi_M}^{-1} \end{aligned}$$

with $k - M \leq \frac{1}{N_s} M_{s-1}$.

Putting this estimation and (3.7) in (3.6), we get,

$$\begin{aligned} |\,\eta_{M,N_1,s}\,|_{-q} &\leq C_q\,[2^{M_s} + (M_{s-1} - M)\, 2^{2k}\, (2N_s+1)^{M_s}]\, {\Pi_M}^{-1} \\ &\leq C_q\,[2^{M_s} + \left(\frac{1}{N_s} M_{s-1}\right) 2^{2k}\, (2N_s+1)^{M_s}]\, {\Pi_M}^{-1} \\ &\leq C_q\,[2^{M_s} + \left(\frac{1}{N_s} M_{s-1}\right) 2^{2M_{s-1}}\, (2N_s+1)^{M_s}]\, {\Pi_M}^{-1} \end{aligned}$$

The estimation for the expression in the brackets is as follows:

$$\begin{aligned} 2^{M_s} + \left(\frac{1}{N_s} M_{s-1}\right) 2^{2M_{s-1}}\, (2N_s+1)^{M_s} &< 2^{M_s} + M_s\, 4^{M_s}\, (3N_s)^{M_s} \\ &= 2^{M_s} + M_s\, (12N_s)^{M_s} \\ &\leq 2^{M_s} + 12^{M_s}\, {N_s}^{2M_s} \\ &< 13^{M_s}\, {N_s}^{2M_s} \end{aligned}$$

So,

$$\begin{aligned} \mid \eta_{M,N_1,s} \mid_{-q} &\leq C_q\,(13)^{M_s}\, {N_s}^{2M_s}\,(2N_s+1)^{M_s}\, \left(\Pi_{q+1}^{M}\, z_k\right)^{-1} \\ &\leq C_q\,(39)^{M_s}\, {N_s}^{2M_s}\, \left(\Pi_{q+1}^{M}\, z_k\right)^{-1} \\ &\leq C_q\,(39)^{M_s}\, {N_s}^{6M_s}\, \left(\Pi_{q+1}^{M}\, z_k\right)^{-1} \\ &\leq C_q\, {l_s}^{-6Q}\, \left(\Pi_{q+1}^{M}\, z_k\right)^{-1} \end{aligned}$$

Hence, as $q \geq p + 6Q + 1$, we have,

$$\begin{aligned} \| e_{M,j,s} \|_p \cdot \mid \eta_{M,N_1 i,s} \mid_{-q} &\leq \left(5\,(2N_{n-u})^p\, M^p\, \Pi_{p+1}^{M} z_k\right) \cdot \left(C_q\, l_s^{-6Q}\, \left(\Pi_{q+1}^{M} z_k\right)^{-1}\right) \\ &\leq 5\, C_q\,(2N)^p\, M^p\, l_s^{-6Q}\, \Pi_{p+1}^{q} z_k \\ &= 5\, C_q\, 2^p\, N^p\, M^p\, l_s^{-6Q}\, z_{p+1} \dots z_q. \end{aligned}$$

Replacing all z_k by l_s we get the bounded sequence on the right as a result of the choice of q, and hence the proof is complete. $\square$

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