MULTI–PERIOD INVENTORY MODELS WITH PRICE PROTECTION

A THESIS
SUBMITTED TO THE DEPARTMENT OF INDUSTRIAL ENGINEERING
AND THE INSTITUTE OF ENGINEERING AND SCIENCE
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
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July, 2007

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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July, 2007

In an environment with declining sales prices, retailers (or any reseller) often face the risk of buying high and selling low. In order to limit their channel partners' exposure to such risks and increase the availability of their products in the marketplace, suppliers often offer price protection. With price protection, a retailer is reimbursed with a percentage of the procurement cost declines, for the inventory that the retailer ordered within a given price protection age limit. We study the optimal inventory policy of the retailer under such price protection terms in a multi–period finite horizon setting with stochastic demand. We propose three different models for the treatment of unsatisfied demand. For the case of full backlogging, we show that the order–up–to type policies are optimal. In a numerical study, we study the behavior of the retailer and investigate the impact of price protection terms on the operational performance of the retailer and the supplier under a variety of settings.

Keywords: Inventory models, price protection, supply chain management, high–tech industry.

ÖZET

FİYAT KORUMASI ALTINDA ÇOK DÖNEMLİ ENVANTER MODELLERİ

Nurdan Ahat
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Temmuz, 2007

Satış fiyatlarının giderek düştüğü bir ortamda, parakendeciler (veya herhangi bir satıcı) bir ürünü yüksek fiyatla alıp daha düşük fiyatla satma riskiyle yüz yüzedir. Tedarikçiler, perakendecilerin bu tür risklere maruz kalmasını engellemek ve ürünlerin pazardaki hazır bulunabilirliğini arttırmak amacıyla, perakendecilere fiyat koruma stratejilerini sunmaktadır. Fiyat koruma stratejileri sayesinde, perakendecinin stoğunda bulunan, belirli bir zaman aralığında ısmarlanan envanterin fiyatında gözlenen düşüşlerin belirli bir yüzdesi perakendeciye tedarikçi tarafından iade edilir. Bu tezde, tedarikçinin perakendeciye belirlenen terimlerle fiyat koruma stratejisi uyguladığı, çok dönemli ve sonlu ufuklu bir ortamda rassal talep altında perakendecinin en iyi envanter kontol politikasının belirlenmesi problemiyle ilgilenilmektedir. Zamanında karşılanamayan talebi farklı şekilde ele alan üç model önerilmektedir. Zamanında karşılanamayan talebin tamamının daha sonra karşılandığı durumlarda seviye esaslı politikaların optimal olduğu gösterilmiştir. Sayısal çalışmalar ile, çeşitli ortamlarda perakendecinin davranışı incelenmiş fiyat koruma politikası koşullarının perakendecinin ve tedarikçinin operasyonel performanslarına etkisi araştırılmıştır.

Anahtar sözcükler: Envanter sistemleri, fiyat koruması, tedarik zinciri yönetimi, yüksek teknoloji endüstrisi.

Acknowledgement

I would like to express my deepest and most sincere gratitude to my advisors Assist. Prof. Dr. Osman Alp and Assist. Prof. Dr. Alper Şen for their invaluable encouragement, trust, guidance and motivating support during my study. They have been supervising me with everlasting patience and interest. Their advices will be the closest, most important and valuable reference for my future.

I would like to express my special thanks to Prof. Dr. Nesim Erkip and Assist. Prof. Dr. Ayşe Kocabıyıkoğlu for accepting to read and review this thesis and for their suggestions.

I am indebted to Prof. Dr. Mustafa Pınar for his invaluable advices and sincere talks and for sharing his enlightening vision with me about my future study and life. I would like to thank Assoc. Prof. Dr. Mustafa Akgül for helping me in solving my computer related problems during my study. I also want to thank all the faculty members for devoting their time and knowledge to me.

I wish to thank my friends Berna Çelebi, Solmaz Şirip, Keziban Akkaya and Çiğdem Akbulut for their existence at the other side of the phone ready to listen and talk to me when the taste of the loneliness started to cause an intolerable pain in my throat. I thank them all, for their luminous glances, closest conversations, and laughters and also, for listening to my delirious ideas with full understanding and tolerance and sharing their valuable thoughts with me.

I would like to thank Ayşegül Altın for her interest and clemency when I was wriggling with an insufferable pain in my stomach on a Sunday morning in Ea331. I would also like to thank Evren Körpeoğlu, Önder Bulut and Ayşegül Altın for their academic assistance and understanding, and being good friends to me.

I am grateful to my dear friends Gülay Samatlı, Esra Aybar, Mehmet Mustafa Tanrıkulu, Sıtkı Gülten, Emre Uzun, Muzaffer Mısırcı, Yahya Saleh, Hatice Çalık, Ali Gökay Erön, Könül Bayramoğlu, and Tuğçe Akbaş. I would like to thank all the graduate students and Kaytarıkçılar for being there.

I would like to thank TÜBİTAK for the financial support they have provided thus make this study happen.

Last but not the least, I would like to express my gratitude to my family for their trust and motivating attitudes. I especially thank to my brother, Mehmet Ahat, for his existence, everlasting love and morale support during my study. He was sometimes my mentor and only source of joy.

And I am grateful to all, who made me to find my way when I fell to the dark wells without any equipment and who made me remember that the stars are still shining and there is still hope...

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Chapter 1

Introduction

The high-tech products are characterized by their short life cycles and high demand variability. The suppliers of these products usually use direct and indirect channels to serve the customers. In the direct channel, the demand is satisfied by the supplier by means of catalogs, mail or internet sales, while in the indirect channel, the demand is satisfied through the distributors, retailers, partners, and etc.

In order to survive in the high-tech market, the supplier needs to offer new products to the market. This reduces the appeal and necessitates reductions in the price of the older products in both direct and indirect channels. Sometimes, suppliers also offer price reductions on older products prior to introduction of new products in order to clear the channel of older inventory. Frequent and significant price declines creates the risk of buying high and selling low for the participants of the indirect channel. Therefore, the indirect channel has less incentive for buying the products at the beginning of their life cycle; this leads to reduced availability of new products in the market. In order to limit the indirect channel's exposure to the risk of buying high and selling low and increase the availability of the product in the market, the manufacturers often offer return or price protection policies.

With price protection, if the supplier reduces the list price of an item, he reimburses the channel participants the difference between what they have paid and the current price for all inventories they have ordered but not yet sold. In some cases, the supplier may impose an age limit on inventories that the channel partners can claim reimbursements, i.e. the protection is valid for only those items that are ordered within a certain period of time. Also, he may impose a limit on the magnitude of the price protection credit, i.e. only a certain percentage of the price change is credited.

Price protection enables the retailer to maintain its profit margin regardless of the price offered by the supplier and reduces the cost of carrying inventory for the retailer. While more price protection is always better for the retailer, the supplier needs to trade off between increased availability of its products in the market and the "cost" of price protection. Price protection can be costly for the supplier in two ways. First type of costs are concrete: with every price decline the manufacturer reimburses the difference between the paid price and current price for the inventory held but not sold in the indirect channel. The second type of costs can be caused by excessive amount of inventory that is carried by the indirect channel due to price protection. This may result in supplier having difficulty in introducing new products to the market as its pipeline is full of older inventory. In addition, price protection limits the supplier's control in tactical pricing. This may be true not only in indirect channel but also in direct channel, as the industry evidence shows that the retail prices in traditional channels are usually same as the prices in supplier's direct channel (See [7] for a few examples). When considering a price drop, the supplier needs to take into account the amount of channel inventory for which needed to be reimbursed under the price protection policy (this is in addition to demand elasticity, competition, amount of component inventory and all other factors that impact a pricing decision). Payments to the channel due to price protection may potentially erase all the benefits that the manufacturer is seeking from a price drop. If not managed properly, price protection costs may add up to significant levels.

Many companies in the high–tech industry adopted price protection policies for their indirect channel. Examples include Hewlett Packard, Apple, Compaq, IBM, Seagate, and Maxtor which offer various price protection and return terms in their contracts to ensure availability of their products in the market ([2], [15], [16], [11], [13]). Hewlett Packard (HP) developed a metric called inventory-driven costs which identifies price protection costs to be one of the major hidden components of cost of holding inventory [2]. Many companies have frequent adjustments to their price protection policies in an effort to strike a balance between keeping their indirect customers happy (hence, the availability of their products in retail) and reducing their price protection costs. Various articles report changes to price protection policies by major companies. Compaq set an age limit of 15 days on price protection [16]. HP announced a reduction in price protection for personal computers [1]. Apple first had set an age limit of 30 days on price protection, but then had to reinstate the full protection after pressure from its indirect customers [15].

In order for the supplier to evaluate any price protection policy alternative, the supplier needs to have a good understanding of how the indirect channel reacts to price changes in the existence of a price protection mechanism. Prerequisite to this is understanding channel behavior under fixed prices and a given price protection mechanism. In this thesis, we study the inventory problem of an indirect channel partner (e.g. a retailer) who faces decreasing prices and is offered a price protection policy in a multi–period setting. The retailer's problem is modeled under the assumption of standard backordering, modified backordering (which allows the loyal customer to buy the item from the newer price), and lost sales. Also, the cost of the price protection policy to the supplier is determined assuming that the retailer is rational and operating with an optimal decision rule.

The organization of the thesis is as follows:

In Chapter 2, we provide a review of the literature in the price protection mechanisms and supply chain coordination.

In Chapter 3, we model the ordering problem of the retailer for three different assumptions for the case of shortages. The first of these is the Standard Backorder Model (SBM), in which the retailer can satisfy the demand later. The customer is charged with the price that is used when the customer first appears. The retailer incurs a backorder cost for each item satisfied with a delay. The second application for the excess demand is the Modified Backorder Model (MBM), in which the retailer satisfies the demand after a while, incurs a backorder cost for each item backordered as in Standard Backorder Model. However this time the owner of the backordered items is charged with the price that is effective when the customer demand is satisfied. The third application for the excess demand is the Lost Sales Model (LSM). In this model excess demand is completely lost. The retailer incurs a shortage cost for each demand lost. The optimal decision rule for the retailer when the supplier offers price protection is determined to be order-up-to type policy for the SBM and MBM.

In Chapter 4, a numerical study with different problem parameters is conducted under MBM and LSM. The optimal decision rule of the retailer is verified to be order-up-to type policy for the MBM and it is observed that the optimal ordering policy for the retailer is also order-up-to type in LSM. The cost paid for the price protection policy by the supplier and the service levels of the retailer is determined when the retailer behaves rationally and applies the optimal order-up-to policy for ordering decisions. The simulation results indicate that the service levels of the retailer increases by introducing price protection. The increment in the service levels decreases as the age of the unsold inventory protected by the supplier increases.

In Chapter 5, we summarize our results and contributions to the literature, and suggest future research directions.

Chapter 2

Literature Survey

In this chapter we study the literature that is closely related to the problem under concern. We begin with the seminal work of Clark and Scarf [3] who model multi-period stochastic inventory problems with dynamic programming. In their paper, a single installation multi-period inventory problem whose objective is to minimize the cost over the horizon is modeled. A positive lead time for the replenishment of the orders is assumed. The state vector at any period is defined as the outstanding orders plus the inventory level of the retailer. A purchasing decision is made at the retailer level after the orders that are placed lead time periods ago are received and they also consider the holding and the shortage costs of the retailer. They assume that the excess demand is backordered. The authors discuss that the optimal decision rule for the retailer is order-up-to type.

In an other stream of research, there are articles which attempt to model the price changes in the multi-period stochastic inventory environments. Gavirneni [6] models the periodic review inventory problem where the product prices change in a Markovian fashion from one period to the next. The product prices decline from one period to the other where the decline structure is defined in a probability transition matrix which is assumed to be regenerative and communicative. It is assumed that there are no set up costs, capacity restrictions, nor lead times. The unsatisfied demand is lost and a linear holding cost for extra inventory on hand is incurred. Under all these assumptions, the author shows that the optimal ordering policy of the retailer is a base stock policy in a single period, finite horizon or discounted infinite horizon problems. Also [4] models the single item, periodic review inventory problem when the demand is stochastic but depends on the the selling prices of the item. They characterize the structure of the optimal pricing and inventory strategy and develop a value iteration method for calculating the optimal strategies. In [5], they consider multiple retailers in the same setting when the demand distribution at each retailer can vary. In the paper, an approximate model for the problem is developed and combined pricing, ordering and allocation strategies are provided.

Another paper that models the price decline of the product in a multi-period environment is by Wang [14]. A single item, single location inventory system that is reviewed periodically and that faces a stochastic demand is considered. The demands of the successive periods are independent and identically distributed. The acquisition cost of the item in the successive periods is modeled as decreasing random variables. Therefore, the selling price of the item decreases in the unit acquisition cost as well. Two pricing fashions are handled in the paper: The retailer price is determined by adding a fixed percent mark-up or fixed amount mark-up to the acquisition cost of the item. It is shown that the order-up-to type policy is optimal for the backorder and the lost sales models.

Price protection problem is first introduced by Lee et al. [8]. They consider a 2-period problem in which the supplier offers the retailer a protection policy. They assume that the manufacturer has no capacity restrictions, the set-up cost is negligible, and the excess demand is lost. They consider two cases for purchasing action of the retailer: the retailer may have a single buying or two buying opportunity. In the single buying opportunity case, it is shown that a properly chosen protection credit coordinates the channel and guarantees a win-win situation. It is shown that the optimal order quantity of the retailer without protection policy is strictly less than the optimal order quantity of the integrated channel with price protection policy.

In the two buying opportunity case, it is shown that the order-up-to type policy is optimal and the order-up-to level of the retailer stays the same with price protection or without price protection in the second period. However, the optimal order quantity of the first period is larger when the supplier offers the retailer a price protection policy. Since the order-up-to level does not change in the second period and the demand is coming from the same distribution in both cases, optimal order quantity in the second period is non-increasing. Therefore, the overall impact of the price protection policy on the retailer's situation is that the total order quantity throughout the horizon increases or stays the same. If the acquisition cost of the item is set in two periods, price protection does not guarantee channel coordination. If the supplier is allowed to adjust the acquisition cost and the protection rebate simultaneously, the assumption that the acquisition cost has to be larger than the manufacturing cost is relaxed, and the acquisition cost of the product is announced at the beginning of the first period, then the channel coordination is restored.

Our model extends the problem that is described in a 2-period environment by Lee et al. [8]. However, we do not examine the channel coordination nor the supplier's problem. We model the retailer's problem in a multi-period environment and analyze the optimal decision rule in the existence of the price protection policy. We also examine the effect of the protection policies to the performance metrics of the players.

Taylor [12] explores the policy combinations that are channel coordinating and implementable that increase the profit margins of the players in the supply chain when compared with the decentralized case. Three channel policies which are commonly adopted in declining price environments for channel coordination are handled. These policies are: price protection (P), end of life returns (E) and mid-life returns (M). The problem is modeled in a 2-period setting with two parties where the demand observed in each period is independent. The stock out and holding costs are negligible and all the exogenous variables are assumed to be known by the players at the beginning of the first period. It is concluded that under the declining price environments, EM achieves channel coordination but does not guarantee a win-win outcome. In the use of PEM, both channel coordination and win-win situation is attained. It is shown that if the retail prices are constant over time, both channel coordination and a win-win situation is achieved by the implementation of EM.

Later, Lu et al. [10] adds rebate (R) policy to E and M which can be utilized instead of P. In R, the supplier specifies a credit which is obtained by the retailer for each unit sold after an acquisition cost decline. The effectiveness of these policies, channel coordination, win-win situation conditions are explored. Lu et al. [10] include the stock-out and holding costs to the model. Similar to the environment of Lee et al. [8], the retailer may have a single buying opportunity or a two buying opportunity. In the single buying opportunity, PEM or REM guarantees a win-win outcome. The conditions under which ME coordinates the channel are determined and discussed. In the two buying opportunity case, it is shown that win-win policy may not exist, and that in the absence of stock-out costs in the first period, PEM always leads to a win-win outcome. The major contribution is that detailed procedures for determining the win-win policy parameters under the assumption that the acquisition cost never exceeds the retailer price are proposed.

Liu [9] explores the effects of the price protection policy on the channel performance in a multi-period environment with deterministic demand structure. In the paper, the age limit of the price protection policy is set to one and the supplier offers full protection to the retailer. The demand is known to the retailer and the supplier and it is decreasing in the selling price. The procurement lead time is negligible and the supplier is ample. Also it is assumed that the market size is non-increasing over time. The performance of the supply chain is compared with the case where there is no protection and supplier uses price-only contract in which the supplier announces the price and the retailer determines the order quantity and selling price in order to satisfy the demand. The impact of the price protection policy to the pricing decisions of the players in the supply chain is studied in the paper. In our study, the effect of the price protection policy to the ordering decision of the retailer and the supplier's performance metrics are explored. Also the problem is modeled in a stochastic environment in decentralized supply chain.

Chapter 3

Model

We consider an inventory control problem of a retailer under a single item, finite planning horizon setting where the supplier with an ample capacity offers a price protection policy to the retailer. The supplier can impose a limit on the price protection credit in the sense that up to a certain portion of the inventory is protected. The supplier sets a certain age limit for the unsold inventory at the retailer to be reimbursed when a price decline decision is made by the supplier. The credit and the age limits do not change throughout the horizon. We consider three different models to treat the excess demand at the retailer: Standard Backorder Model (SBM), Modified Backorder Model (MBM), and Lost Sales Model (LSM). The set-up cost is assumed to be zero so that the retailer is not charged any additional amount for placing an order to the supplier. The retailer may be subject to a procurement lead time where the items ordered are received after a period of lead time. The holding and shortage costs are incurred after observing the demand and the acquisition cost is incurred at the beginning of the period. In SBM and MBM any excess demand is backordered with a cost incurred per item per period, whereas in LSM excess demand is lost with a cost incurred for each unit lost. In SBM and MBM, the selling price charged to a backordered customer is different. In SBM, at the time of the demand realization, the customers whose demands are backordered pay the current selling price of the item but receive their demand when the retailer's stocks become available. Since the selling price of the item may drop in time in our problem environment, the price that has already been charged to a backordered customer may be higher than the price of the item at the time of the backorder clearance. The difference between the selling prices can be interpreted as the reservation cost that is incurred by the backordered customer for the motivation of the retailer to order more and keep the item in his stocks. In MBM, the backordered customers pay the selling price of the item that is effective when they receive their demand from the retailer. In LSM, the excess demand is lost. The retailer incurs a shortage cost and the potential profit from this customer is also lost.

The sequence of the events at any period at the retailer is as follows:

1. At the end of a period, the inventory level is reviewed and determined. If the inventory level is less than zero, shortage costs are incurred. If the inventory level is positive, then the excess inventory is carried to the next period by incurring a holding cost.

2. At the beginning of the next period, if there is a reduction is the acquisition costs and there exists protected inventory at the retailer then the supplier reimburses the protected quantity.

3. An order is placed to by the retailer based on the current inventory position (i.e. inventory level plus the orders placed but not received yet). The acquisition costs are incurred.

4. The order which was placed lead time periods ago is received and the inventory level is updated.

5. Demand is observed.

At the end of the planning horizon, if there is positive inventory left at the retailer and in case a decline in the purchasing price, the unsold inventory under protection is reimbursed by the supplier in all models. In SBM and MBM, the excess demand at the last period of the horizon is satisfied by the retailer by means of market clearance obligation of the retailer. However in LSM, it is lost.

Before detailing the description of the model, we provide the notation that is used throughout the thesis:

N	=	Number of periods,
a	=	Protection age limit of the price protection policy,
α	=	Price protection credit, $0 < \alpha \leq 1$,
l	=	Procurement lead time between the retailer and the supplier,
W_k	=	Random variable denoting the demand during the k^{th} period,
$F(w_k)$	=	The distribution function of the demand, W_k,
$f(w_k)$	=	The probability distribution function of the demand, W_k,
$\overline{D}_k$	=	Average demand at the k^{th} period,
γ	=	Discount factor,
b_k	=	Unit shortage cost in period k,
h_k	=	Unit holding cost in period k,
c_k	=	Unit acqusition cost of the item in period k,
Δc_k	=	$c_k - c_{k-1}$,
p_k	=	Unit selling price of the item in period k^{th},
Δp_k	=	$p_k - p_{k-1}$,
q_k	=	Order placed by the retailer at the beginning of the k^{th} period,
x_k	=	Inventory level at the beginning of the k^{th} period after, the replenishment
$\tilde{x}_k$	=	State of the system at the beginning of the k^{th} period including the starting inevntory level, x_k, and the orders placed in the past a periods,
$J_k(\tilde{x}_k)$	=	Minimum cost incurred in periods k, $k+1, ..., N$ in SBM, when the system state is $\tilde{x}_k$,
$R_k(\tilde{x}_k)$	=	Maximum profit obtained in periods k, $k+1, ..., N$ in MBM and LSM, when the system state is $\tilde{x}_k$.

At the beginning of period k, we assume that c_k's and p_k's are known by the channel partners, $p_k > c_k$, and $a \geq l$

In Sections 3.1, 3.2 and 3.3, we present Dynamic Programming (DP) models for SBM, MBM and LSM respectively. Initially, we provide the common arguments that are used in the models. In order to construct a DP model, we need to define the state vector, $\tilde{x}_k$, that contains the state variables. The state variables of the problem at any period are the orders that the retailer placed in the last a periods and the on hand inventory at the beginning of the period. The state vector for the price protection problem is the following:

$$\tilde{x}_k = \Big(\; x_k, \; q_{k-a}, \; q_{k-a+1}, ..., q_{k-l}, \; q_{k-l+1}...q_{k-1} \; \Big).$$

The next step is determining the evolution equation that is used to determine the following state vector given the current state vector. The state vector $\tilde{x}_k$ at period k, evolves into the following state vector at period $k+1$ at SBM and MBM.

$$\tilde{x}_{k+1} = \Big(\; x_k + q_{k-l} - w_k, \; q_{k-a+1}, \; q_{k-a+2}, ..., q_{k-l}, \; q_{k-l+1}, ..., q_{k-1}, \; q_k \; \Big).$$

There is a small change in the evolution equation in the LSM which is discussed in Section 3.3.

Let $v_k = q_{k-l+1} + ... + q_{k-1}$ be the outstanding orders at the k^{th} period after receiving the order placed l periods ago. Thus, the total protected amount in period k, η_k, is given by:

$$\eta_k = \begin{cases} \alpha(q_{k-a} + ... + q_{k-l} + v_k) & \text{if } x_k - q_{k-a} - ... - q_{k-l} - v_k \geq 0 \\ \alpha(x_k + v_k) & \text{if } x_k - q_{k-a} - ... - q_{k-l} - v_k < 0 \\ 0 & \text{if } x_k + v_k \leq 0 \end{cases}$$

If $x_k - q_{k-a} - ... - q_{k-l} - v_k \geq 0$, the retailer has still items unsold in his inventory that are purchased a periods ago. The protected quantity is given by: $\alpha(q_{k-a} + ... + q_{k-l} + v_k)$ which is the maximum possible inventory under protection.

If $x_k - q_{k-a} - ... - q_{k-l} - v_k < 0$, the retailer has already started to sell the items that are purchased a periods ago and in this case the inventory under protection is: $\alpha(x_k + v_k)$.

If $x_k + v_k \leq 0$, there are no unsold items in the retailer's stocks or in the pipeline and the retailer has already sold all the inventory to the customers and therefore the inventory under protection is zero.

Therefore, total number of the protected items in the k^{th} period is given by

$$\eta_k = \alpha \min\{q_{k-a} + ... + q_{k-l} + v_k, x_k + v_k\}^+.$$

3.1 Standard Backorder Model

In SBM, since the price charged to the backordered customers is always the effective price at the time of demand realization. The ordering policy employed does not have any impact on the total revenue generated from the customers. Hence we can model this environment with a cost minimization objective. Let $J_k(\tilde{x}_k)$ be the minimum cost incurred in period k when the system state is $\tilde{x}_k$, then

SBM-DP:

$$\begin{aligned}
J_{N+l+1}(\tilde{x}_{N+l+1}) &= \alpha \Delta c_{N+l+1} \min\{q_{N+l+1-a} + ... + q_N, x_{N+l+1}\}^+ \\
&\quad + c_{N+l+1} \max(-x_{N+l+1}, 0) \\
J_k(\tilde{x}_k) &= \min_{q_k=0}\{c_k q_k + L(x_k + q_{k-l}) \\
&\quad + \alpha(c_k - c_{k-1}) \min\{q_{k-a} + ... + q_{k-l} + v_k, x_k + v_k\}^+ \\
&\quad + \gamma E_{W_k}(J_{k+1}(\tilde{x}_{k+1}))\} \;\; \forall \;\; k = N+1, ..., N+l \\
J_k(\tilde{x}_k) &= \min_{q_k \geq 0}\{c_k q_k + L(x_k + q_{k-l}) \\
&\quad + \alpha(c_k - c_{k-1}) \min\{q_{k-a} + ... + q_{k-l} + v_k, x_k + v_k\}^+ \\
&\quad + \gamma E_{W_k}(J_{k+1}(\tilde{x}_{k+1}))\} \\
&\quad \forall \; k = 1, 2, ..., N, \;\; F_{W_k}(0) = 1 \;\; \forall \; k = 1, ..., l
\end{aligned}$$

where $L(y) = h_k \int_0^y (y - w_k) dF(w_k) + b_k \int_y^\infty (w_k - y) dF(w_k)$

In the above formulation if $k - l < 0$, $q_{k-l} \equiv 0$. The retailer stops ordering at the N^{th} period but the retailer continues to observe demand in periods $N + 1$, $N + 2, ..., N + l$. Hence, even though the end of the planning horizon is N, the business transactions finish at the end of period $(N + l)$. We assume that the unit acquisition cost is the same throughout the periods $N + 1$, $N + 2, ..., N + l$ so that $c_{N+1} = c_k \;\; \forall \; k = N + 1, ..., N + l$.

If the procurement lead time is zero and the protection age limit is one, the DP is written as follows:
SBM-DP-1:

$$\begin{aligned} J_{N+1}(\tilde{x}_{N+1}) &= \alpha \Delta c_{N+1} \min(x_{N+1}, q_N)^+ + c_{N+1} \max(-x_{N+1}, 0) \\ J_k(\tilde{x}_k) &= \min_{q_k \geq 0} \{c_k q_k + L(x_k + q_k) + \alpha \Delta c_k \min\{x_k, q_{k-l}\}^+ \\ &+ \gamma E_{W_k} \left(J_{k+1}(\tilde{x}_{k+1})\right)\} \ \forall\ k = 1, 2, ..., N. \end{aligned}$$

In Theorem 1, for $a = 1$ and $l = 0$ we show that SBM-DP-1 can be transformed into another DP formulation, where the inventory level attained after ordering, y_k, is the only decision variable, the optimal of this variable is found by minimizing a convex function which is independent of the state variables and hence an order-up-to type policy is the optimal ordering policy of the retailer under the condition that the total holding and backorder costs at any period is greater than the protection credit paid by the supplier to the retailer (i.e. $h_k + b_k + \alpha \Delta c_{k+1} \geq 0$). According to the the order-up-to type policy, the retailer is supposed to increase the inventory level to an optimal order-up-to point that is determined by the minimum of a convex function, if the inventory level of the retailer is less than the optimal-order-up-to point. Otherwise, the retailer does not order.

$$q_k^* = \begin{cases} y_k^* - x_k & \text{if } x_k < y_k^* \\ 0 & \text{if } x_k \geq y_k^* \end{cases}$$

Also, optimal order-up-to level in the last period is given by

$$y_N^* = F^{-1} \left(\frac{b_N - c_N + \gamma c_{N+1}}{h_N + b_N + \gamma \alpha \Delta c_{N+1} + \gamma c_{N+1}} \right),$$

if total holding, backorder and selling price of the item is greater than the protection credit offered by the supplier in the last period (i.e. $h_N + b_N + \gamma \alpha \Delta c_{N+1} + \gamma c_{N+1} \geq 0$).

Theorem 1 *For $a = 1$, $l = 0$, if $h_k + b_k + \alpha \Delta c_{k+1} \geq 0\ \forall\ k = 1, 2, ..., N-1$, and if $k = N$ $h_N + b_N + \gamma \alpha \Delta c_{N+1} + \gamma c_{N+1} \geq 0$, then,* *(i) SBM-DP-1 is equivalent to the following problem:*

$$\begin{aligned} J_{N+1}(\tilde{x}_{N+1}) &= \alpha\Delta c_{N+1}\min(x_{N+1}, q_N)^+ + c_{N+1}\max(-x_{N+1}, 0) \\ J_k(\tilde{x}_k) &= \min_{y_k \geq x_k}\{G_k(y_k)\} - \gamma\alpha\Delta c_{k+1}\Phi(x_k) + \alpha\Delta c_k \min(x_k, q_{k-1})^+ - c_k x_k \\ &\quad + \left(\sum_{i=k}^{i=N-1} \gamma^{i+1-k}\left(c_{i+1}\overline{D}_i + G_{i+1}(y^*_{i+1})\right)\right) + \gamma^{N+1-k}c_{N+1}\overline{D}_N \\ &\quad \forall\, k = 1, 2, ..., N \end{aligned}$$

where

$$\begin{aligned} G_k(y_k) &= (c_k - \gamma c_{k+1})y_k + L(y_k) + \gamma\int_0^{y_k - y^*_{k+1}} G_{k+1}(y_k - w_k)dF(w_k) \\ &\quad - \gamma G_{k+1}(y^*_{k+1})F(y_k - y^*_{k+1}) - \gamma^2\alpha\Delta c_{k+2}\int_0^{\infty}\Phi(y_k - w_k)dF(w_k) \\ &\quad + \gamma\alpha\Delta c_{k+1}\Phi(y_k)\forall\, k = 1, 2, ..., N-1. \end{aligned}$$

$$G_N(y_N) = (c_N - \gamma c_{N+1})y_N + L(y_N) + \gamma\Phi(y_N)(\alpha\Delta c_{N+1} + c_{N+1})\ for\ k = N.$$

$\Phi(y) = \int_0^y F(t)dt$, $G_{N+1}(y_{N+1}) = 0$, *and*
$y^*_k = \text{argmin}_{y_k}\{G_k(y_k)\}\ \forall\, k = 1, 2, ..., N.$
(ii) $J_k(\tilde{x}_k)$ *is convex in* $\tilde{x}_k$ *and* $G_k(y_k)$ *is convex in* y_k $\forall\, k = 1, 2, ..., N$.
(iii) Optimal ordering policy of the retailer is an order-up-to type policy that states if the inventory level of the retailer is less than y^*_k*, the order quantity bring the inventory level to* y_k**, otherwise do not order. The following summarizes the order-up-to type policy,*

$$q^*_k = \begin{cases} y^*_k - x_k & if\ x_k < y^*_k \\ 0 & if\ x_k \geq y^*_k \end{cases}$$

(iv) Optimal order-up-to level of the last period is:

$$y^*_N = F^{-1}\left(\frac{b_N + \gamma c_{N+1} - c_N}{h_N + b_N + \gamma\alpha\Delta c_{N+1} + \gamma c_{N+1}}\right).$$

□

If the procurement lead time is zero and the protection age limit is 2, the DP can be written as follows:
SBM-DP-2:

$$\begin{aligned} J_{N+1}(\tilde{x}_{N+1}) &= \alpha\Delta c_{N+1}\min(x_{N+1}, q_N + q_{N-1})^+ + c_{N+1}\max(-x_{N+1}, 0) \\ J_k(\tilde{x}_k) &= \min_{q_k\geq 0}\{c_k q_k + L(x_k + q_k) + \alpha\Delta c_k \min\{x_k, q_{k-1} + q_{k-2}\}^+ \\ &+ \gamma E_{W_k}\left(J_{k+1}(\tilde{x}_{k+1})\right)\} \;\; \forall \;\; k = 1, 2, ..., N \end{aligned}$$

In Theorem 2, We extend the results for Theorem 1 to $a = 2$ when $l = 0$.

Theorem 2 *For $a = 2$, $l = 0$, if $h_k + b_k + \gamma\alpha\Delta c_{k+1} \geq 0 \;\forall\; k = 1, 2, ..., N-1$, and if $k = N$ $h_N + b_N + \gamma\alpha\Delta c_{N+1} + \gamma c_{N+1} \geq 0$,*
(i) SBM-DP-2 is equivalent to the following problem:

$$J_{N+1}(\tilde{x}_{N+1}) = \alpha\Delta c_{N+1}\min(x_{N+1}, q_N + q_{N-1})^+ + c_{N+1}\max(-x_{N+1}, 0)$$

$$\begin{aligned} J_N(\tilde{x}_N) &= \min_{y_N \geq x_N}\{G_N(y_N)\} - \gamma\alpha\Delta c_{N+1}\Phi(x_N - q_{N-1}) + \gamma c_{N+1}\overline{D}_N \\ &+ \alpha\Delta c_N \min(x_N, q_{N-1} + q_{N-2})^+ - c_N x_N. \end{aligned}$$

where

$$G_N(y_N) = (c_N - \gamma c_{N+1})y_N + L(y_N) + \gamma\Phi(y_N)(\alpha\Delta c_{N+1} + c_{N+1}).$$

$$\begin{aligned} J_k(\tilde{x}_k) &= \min_{y_k \geq x_k}\{G_k(y_k)\} - \gamma^2\alpha\Delta c_{k+2}\int_0^\infty \Phi(x_k - w_k)dF(w_k) \\ &- \gamma\alpha\Delta c_{k+1}\Phi(x_k - q_{k-1}) \\ &+ \alpha\Delta c_k \min(x_k, q_{k-2} + q_{k-1})^+ - c_k x_k \\ &+ \left(\sum_{i=k}^{i=N-1} \gamma^{i+1-k}\left(c_{i+1}\overline{D}_i + G_{i+1}(y^*_{i+1})\right)\right) + \gamma^{N+1-k}c_{N+1}\overline{D}_N \\ &\forall\, k = 1, 2, ..., N-1. \end{aligned}$$

$$\begin{aligned} G_k(y_k) &= (c_k - \gamma c_{k+1})y_k + L(y_k) + \gamma\int_0^{y_k - y^*_{k+1}} G_{k+1}(y_k - w_k)dF(w_k) \\ &- \gamma G_{k+1}(y^*_{k+1})F(y_k - y^*_{k+1}) \\ &- \gamma^3\alpha\Delta c_{k+3}\int_0^\infty\int_0^\infty \Phi(y_k - w_{k+1} - w_k)dF(w_{k+1})dF(w_k) + \gamma\alpha\Delta c_{k+1}\Phi(y_k) \\ &\; if\;\; k = 1, 2, ..., N-2, \end{aligned}$$

$$
\begin{aligned}
G_{N-1}(y_{N-1}) &= (c_{N-1} - \gamma c_N) y_{N-1} + L(y_{N-1}) + \gamma \int_0^{y_{N-1}-y_N^*} G_N(y_{N-1} - w_{N-1}) dF(w_{N-1}) \\
&\quad - \gamma G_N(y_N^*) F(y_{N-1} - y_N^*) + \gamma \alpha \Delta c_N \Phi(y_{N-1}) \ \ if k = N-1.
\end{aligned}
$$

$\Phi(y) = \int_0^y F(t)dt$,
$G_{N+1}(y_{N+1}) = 0$, $y^*_{N+1} = 0$,
$y^*_k = \text{argmin}_{y_k}\{G_k(y_k)\} \ \forall\, k = 1, 2, ..., N$.
(ii) *$J_k(\tilde{x}_k)$ is convex in $\tilde{x}_k$ and $G_k(y_k)$ is convex in y_k $\forall\, k = 1, 2, ..., N$.*
(iii) *Optimal ordering policy of the retailer is an order-up-to type policy. That is,*

$$
q_k^* = \begin{cases} y_k^* - x_k & if\ x_k < y_k^* \\ 0 & if\ x_k \geq y_k^* \end{cases}
$$

(iv) *The order-up-to level of the last period is*

$$
y_N^* = F^{-1}\left(\frac{b_N - c_N + \gamma c_{N+1}}{h_N + b_N + \gamma \alpha \Delta c_{N+1} + \gamma c_{N+1}}\right).
$$

□

In Theorems 1 and 2, we prove that the base stock policy is optimal. According to the base stock policy, the retailer is supposed to increase the inventory level to an optimal order-up-to point that is determined by the minimum of a convex function, if the inventory level of the retailer is less than the optimal-order-up-to point. Otherwise, the retailer does not order. Also, we observe that the order-up-to levels of the models with single protection period and double protection period are the same for the last period. This can be the consequence of allowing backorders. In this case no demand is lost, some of them are satisfied on time and some of them are satisfied with a little delay and a backorder cost. Thus the market share remains the same for the retailer. In this setting increasing the protection period does not affect the ordering amounts but the total costs. Also, we observe the convexity conditions for the single and double period protection is the same. In both cases, the convexity is preserved at any intermediate period if the total holding and backorder costs of the retailer is higher than the protection credit offered by the supplier. That is if the protection credit offered by the supplier is higher than the total holding and backorder costs incurred by the retailer, then the retailer will simply prefer no action with the customers and the model will not be valid. Also, for the last period in order to preserve the convexity, total holding, backorder and discounted acquisition cost of the item for the ending period $(N + 1)$ should be grater or equal to the discounted protection credit offered by the supplier for the ending period $(N+1)$. Otherwise, the retailer would choose no action with the customers and the model would be meaningless.

SBM is applicable for the products whose demand rate is high but manufacturing rate is low. Therefore, the customers who cannot find the product in the market becomes willing to pay the reservation cost and wait for the product to become available in the retailer's stocks. Reservation cost can be considered as the difference in the selling price of the product between the period that the demand is observed and the period that it is satisfied. Since the acquisition cost declines in time, reservation cost is non-negative. Modeling the environments that includes the products like computer game consoles (e.g. Sony Play Station) using SBM would be more meaningful. Since these products are manufactured in limited quantity and the demand of the item depend on the popularity of the product among the customers. Also, there is a decline in the acquisition cost of the product, since the newer ones are prepared for the market. Therefore, the customers are willing to pay the reservation cost for the product that they want to have.

3.2 Modified Backorder Model

In MBM, the selling price charged to a customer whose demand is backordered is the price that is effective at the time of the backorder clearance. The total expected profit obtained in such environments depends on the operating policies employed, since the selling price charged to a backordered customer may drop in the mean time until the backorder is cleared. Hence, problem in this setting should be modeled with a profit maximization objective. The following formulation is valid for the profit to go function in the k^{th} period:

MBM-DP:

$$
\begin{aligned}
R_{N+l+1}(\tilde{x}_{N+l+1}) &= -\alpha\Delta c_{N+1}\min\{q_{N+l+1-a}+...q_N, x_{N+l+1}\}^{+} \\
&\quad + (p_{N+1}-c_{N+1})\max(-x_{N+l+1},0) \\
R_k(\tilde{x}_k) &= \max_{q_k=0}\{-c_kq_k + p_k\left(\max(0,\min(0,x_k+q_{k-l})-x_k)\right. \\
&\quad \left. + E_{W_k}\{\min(x_k+q_{k-l},W_k)^{+}\}\right) \\
&\quad - L(x_k+q_{k-l}) \\
&\quad - \alpha(c_k-c_{k-l})\min\{q_{k-a}+...+q_{k-l-1}+v_k, x_k+v_k\}^{+} \\
&\quad + \gamma E_{W_k}(R_{k+1}(\tilde{x}_{k+1}))\} \;\; \forall\, k=N+1,...,N+l, \\
R_k(\tilde{x}_k) &= \max_{q_k\geq 0}\{-c_kq_k + p_k\left(\max(0,\min(0,x_k+q_{k-l})-x_k)\right. \\
&\quad \left. + E_{W_k}\{\min(x_k+q_{k-l},W_k)^{+}\}\right) \\
&\quad - L(x_k+q_{k-l}) \\
&\quad - \alpha(c_k-c_{k-l})\min\{q_{k-a}+...+q_{k-l-1}+v_k, x_k+v_k\}^{+} \\
&\quad + \gamma E_{W_k}(R_{k+1}(\tilde{x}_{k+1}))\} \\
&\quad \forall\, k=1,2,...,N,
\end{aligned}
$$

where

$$
\begin{aligned}
L(y) &= h_k\int_0^y (y-w_k)dF(w_k) + b_k\int_y^\infty (w_k-y)dF(w_k) \\
& F_{W_k}(0)=1 \;\; \forall\, k=1,...,l.
\end{aligned}
$$

Let U_k be the money collected by the retailer at the end of the k^{th} period provided that the beginning net inventory level of the retailer is x_k, the inventory delivered to the retailer's stocks at the beginning of the k^{th} period is q_{k-l}, and the order placed by the retailer after the replenishment is q_k. Then we have

$$
U_k = \begin{cases} p_kq_{k-l} & \text{if } x_k+q_{k-l}<0 \text{ and } x_k<0 \\ -p_kx_k + E_{W_k}(\min(x_k+q_{k-l},W_k)) & \text{if } x_k+q_{k-l}\geq 0 \text{ and } x_k<0 \\ p_kE_{W_k}(\min(x_k+q_{k-l},W_k)) & \text{if } x_k+q_{k-l}\geq 0 \text{ and } x_k\geq 0 \end{cases}
$$

This is exactly obtained by the following expression

$$
p_k\left(\max(0,\min(0,x_k+q_{k-l})-x_k) + E_{W_k}\{\min(x_k+q_{k-l},W_k)^{+}\}\right).
$$

For $l = 0$ and $a = 1$ we have,
MBM-DP-1:

$$
\begin{aligned}
R_{N+1}(\tilde{x}_{N+1}) &= -\alpha \Delta c_{N+1} \min(x_{N+1}, q_N)^+ + (p_{N+1} - c_{N+1}) \max(-x_{N+1}, 0) \\
R_k(\tilde{x}_k) &= \max_{q_k \geq 0} \{-c_k q_k + p_k \left(\max(0, \min(0, x_k + q_k) - x_k)\right. \\
&\quad + \left. E_{W_k}\{\min(x_k + q_k, W_k)^+\}\right) \\
&\quad - L(x_k + q_k) - \alpha \Delta c_k \min(x_k, q_{k-1})^+ + \gamma E_{W_k}(R_{k+1}(\tilde{x}_{k+1}))\} \\
&\quad \forall \;\; k = 1, 2, ..., N
\end{aligned}
$$

In Theorem 3, for $a = 2$ and $l = 0$ we show that MBM-DP-1 can be transformed into another DP formulation, where the inventory level attained after ordering, y_k, is the only decision variable, the optimal of this variable is found by maximizing a concave function which is independent of the state variables and hence order-up-to type policy is optimal ordering policy for the retailer under the condition that the total holding, backorder and selling price of the item in the $k'th$ period is higher than the discounted total of the protection credit that will be delivered and the selling price of the item in the next period (i.e. $(-p_k+\gamma p_{k+1}-\gamma\alpha\Delta c_{k+1}-h_k-b_k) \leq 0$). According to the order-up-to type policy, the retailer is supposed to increase the inventory level to an optimal order-up-to point if the inventory level of the retailer is less than this point. Otherwise, the retailer does not order. More formally, the following is valid for the order-up-to type policy:

$$
q_k^* = \begin{cases} y_k^* - x_k & \text{if } x_k < y_k^* \\ 0 & \text{if } x_k \geq y_k^* \end{cases}
$$

Also, optimal order-up-to level in the last period is given by

$$
y_N^* = F^{-1} \left(\frac{b_N + c_N - p_N - \gamma c_{N+1} + \gamma p_{N+1}}{-p_N - \gamma \alpha \Delta c_{N+1} + \gamma p_{N+1} - \gamma c_{N+1} - h_N - b_N} \right),
$$

if the total holding, backorder and selling price of the item is higher than the discounted total of the protection credit that will be paid to the retailer at the end of the horizon and the profit margin of the retailer that will be obtained from selling a single item to a customer (i.e. $(-p_N+\gamma p_{N+1}-\gamma\alpha\Delta c_{N+1}-\gamma c_{N+1}-h_N-b_N)\le 0$).

Theorem 3 *For $a=1$, $l=0$, if $(-p_k+\gamma p_{k+1}-\gamma\alpha\Delta c_{k+1}-h_k-b_k)\le 0\ \forall\ k=1,2,...,N-1$, and if $(-p_N+\gamma p_{N+1}-\gamma\alpha\Delta c_{N+1}-\gamma c_{N+1}-h_N-b_N)\le 0$ then, (i) assuming that $y_k=x_k+q_k$ and $y_k>0$, therefore* $\max(0,\min(0,x_k+q_k)-x_k)=\max(-x_k,0)$
MBM-DP-1 is equivalent to the following problem:

$$\begin{aligned}R_N(\tilde{x}_N) &= \max_{y_N\ge x_N}\{G_N(y_N)\}+\gamma\alpha\Delta c_{N+1}\Phi(x_N)+\gamma(p_{N+1}-c_{N+1})\overline{D}_N\\ &\quad + p_N\max(-x_N,0)+c_Nx_N-\alpha\Delta c_N\min(x_N,q_{N-1})^+\end{aligned}$$

where

$$\begin{aligned}G_N(y_N) &= y_N(-c_N+p_N+\gamma c_{N+1}-\gamma p_{N+1})\\ &\quad +\Phi(y_N)(-p_N-\gamma\alpha\Delta c_{N+1}+\gamma p_{N+1}-\gamma c_{N+1})-L(y_N),\end{aligned}$$

$$\begin{aligned}R_k(\tilde{x}_k) &= \max_{y_k\ge x_k}\{G_k(y_k)\}+\gamma\alpha\Delta c_{k+1}\Phi(x_k)\\ &\quad + p_k\max(-x_k,0)-\alpha\Delta c_k\min(x_k,q_{k-1})^+-c_kx_k\\ &\quad + \left(\sum_{i=k}^{i=N-1}\gamma^{i+1-k}\left((p_{i+1}-c_{i+1})\overline{D}_i+G_{i+1}(y^*_{i+1})\right)\right)+\gamma^{N+1-k}c_{N+1}\overline{D}_N\\ &\quad \forall\, k=1,2,...,N-1\end{aligned}$$

where

$$\begin{aligned}G_k(y_k) &= (-c_k+\gamma c_{k+1}+p_k)y_k+(-p_k+\gamma p_{k+1}-\gamma\alpha\Delta c_{k+1})\Phi(y_k)\\ &\quad - L(y_k)+\gamma\int_0^{y_k-y^*_{k+1}}G_{k+1}(y_k-w_k)dF(w_k)-\gamma G_{k+1}(y^*_{k+1})F(y_k-y^*_{k+1})\\ &\quad + \gamma^2\alpha\Delta c_{k+2}\int_0^{\infty}\Phi(y_k-w_k)dF(w_k).\end{aligned}$$

$\Phi(y)=\int_0^y F(t)dt$,
$G_{N+1}=0$, $y^*_{N+1}=0$, $x_k+q_k\ge 0$,
$y^*_k=\text{argmin}_{y_k}\{G_k(y_k)\}\ \forall\, k=1,2,...,N$.
(ii) $R_k(\tilde{x}_k)$ is concave in $\tilde{x}_k$ and q_{k-1} and $G_k(y_k)$ is concave in $y_k\ \forall\, k=1,2,...,N$.

(*iii*) *Optimal ordering policy of the retailer is an order-up-to type policy,*

$$q_k^* = \begin{cases} y_k^* - x_k & \text{if } x_k < y_k^* \\ 0 & \text{if } x_k \geq y_k^* \end{cases}$$

(*iv*) *The optimal order-up-to level in the last period is:*

$$y_N^* = F^{-1}\left(\frac{b_N + c_N - p_N - \gamma c_{N+1} + \gamma p_{N+1}}{-p_N - \gamma\alpha\Delta c_{N+1} + \gamma p_{N+1} - \gamma c_{N+1} - h_N - b_N}\right).$$

□

For $a = 2$ and $l = 0$, we have the following formulation:
MBM-DP-2:

$$\begin{aligned} R_{N+1}(\tilde{x}_{N+1}) &= -\alpha\Delta c_{N+1}\min(x_{N+1}, q_N + q_{N-1})^+ \\ &\quad + (p_{N+1} - c_{N+1})\max(-x_{N+1}, 0) \\ R_k(\tilde{x}_k) &= \max_{q_k \geq 0}\{-c_k q_k + p_k\left(\max(0, \min(0, x_k + q_k) - x_k)\right. \\ &\quad \left. + E_{W_k}\{\min(x_k + q_k, W_k)^+\}\right) \\ &\quad - L(x_k + q_k) - \alpha\Delta c_k \min\{x_k, q_{k-1} + q_{k-2}\}^+ \\ &\quad + \gamma E_{W_k}(R_{k+1}(\tilde{x}_{k+1}))\} \\ &\quad \forall\ k = 1, 2, ..., N. \end{aligned}$$

In Theorem 4, we extend the results of Theorem 3 to $a = 2$ case.

Theorem 4 *For $a = 2$, $l = 0$, if $(-\gamma\alpha\Delta c_{k+1} + \gamma p_{k+1} - p_k - h_k - b_k) \leq 0$ $\forall\ k = 1, 2, ..., N-1$, and if $k = N$ $(-p_N - \gamma\alpha\Delta c_{N+1} + \gamma(p_{N+1} - c_{N+1}) - h_N - b_N) \leq 0$ then,*
(*i*) *assuming that $y_k = x_k + q_k$ and $y_k > 0$, $\max(0, \min(0, x_k + q_k) - x_k) = \max(-x_k, 0)$ MBM-DP-2 is equivalent to the following problem:*

$$\begin{aligned} R_N(\tilde{x}_N) &= \max_{y_N \geq x_N}\{G_N(y_N)\} + \gamma\alpha\Delta c_{N+1}\Phi(x_N - q_{N-1}) + p_N \max(-x_N, 0) \\ &\quad + c_N x_N - \alpha\Delta c_N \min(x_N, q_{N-1} + q_{N-2})^+ + \gamma(p_{N+1} - c_{N+1})\overline{D}_N \end{aligned}$$

where

$$\begin{aligned} G_N(y_N) &= y_N(-c_N + p_N - \gamma(p_{N+1} - c_{N+1})) \\ &\quad + \Phi(y_N)(-p_N - \gamma\alpha\Delta c_{N+1} + \gamma(p_{N+1} - c_{N+1})) - L(y_N), \end{aligned}$$

$$\begin{aligned} R_{N-1}(\tilde{x}_{N-1}) &= \max_{y_{N-1} \geq x_{N-1}} \{G_{N-1}(y_{N-1})\} \\ &\quad + \gamma^2\alpha\Delta c_{N+1} \int_0^\infty \Phi(x_{N-1} - w_{N-1}) dF(w_{N-1}) \\ &\quad + \gamma\alpha\Delta c_N \Phi(x_{N-1} - q_{N-2}) \\ &\quad + p_{N-1}\max(-x_{N-1}, 0) - \alpha\Delta c_{N-1}\min(x_{N-1}, q_{N-2} + q_{N-3})^+ \\ &\quad + c_{N-1}x_{N-1} + \gamma(p_N - c_N)\overline{D}_{N-1} + \gamma^2(p_{N+1} - c_{N+1})\overline{D}_N + \gamma G_N(y_N^*) \end{aligned}$$

where

$$\begin{aligned} G_{N-1}(y_{N-1}) &= (-c_{N-1} + \gamma c_N + p_{N-1})y_{N-1} \\ &\quad + (-\gamma\alpha\Delta c_N + \gamma p_N - p_{N-1})\Phi(y_{N-1}) \\ &\quad - L(y_{N-1}) + \gamma \int_0^{y_{N-1} - y_N^*} G_N(y_{N-1} - w_{N-1}) dF(w_{N-1}) \\ &\quad - \gamma G_N(y_N^*) F(y_{N-1} - y_N^*), \end{aligned}$$

$$\begin{aligned} R_k(\tilde{x}_k) &= \max_{y_k \geq x_k} \{G_k(y_k)\} \\ &\quad + \gamma^2\alpha\Delta c_{k+2} \int_0^\infty \Phi(x_k - w_k) dF(w_k) \\ &\quad + \gamma\alpha\Delta c_{k+1}\Phi(x_k - q_{k-1}) + p_k\max(-x_k, 0) \\ &\quad - \alpha\Delta c_k \min(x_k, q_{k-2} + q_{k-1})^+ + c_k x_k \\ &\quad + \left(\sum_{i=k}^{i=N-1} \gamma^{i+1-k} \left((p_{i+1} - c_{i+1})\overline{D}_i + G_{i+1}(y_{i+1}^*) \right) \right) + \gamma^{N+1-k} c_{N+1} \overline{D}_N \end{aligned}$$

where

$$\begin{aligned} G_k(y_k) &= (-c_k + \gamma c_{k+1} + p_k)y_k + (-\gamma\alpha\Delta c_{k+1} + \gamma p_{k+1} - p_k)\Phi(y_k) \\ &\quad - L(y_k) + \gamma \int_0^{y_k - y_{k+1}^*} G_{k+1}(y_k - w_k) dF(w_k) \\ &\quad - \gamma G_{k+1}(y_{k+1}^*) F(y_k - y_{k+1}^*) \\ &\quad + \gamma^3\alpha\Delta c_{k+3} \int_0^\infty \int_0^\infty \Phi(y_k - w_k - w_{k+1}) dF(w_k) dF(w_{k+1}) \Big\}. \end{aligned}$$

$\Phi(y) = \int_0^y F(t)dt$,
$G_{N+1} = 0$, $y^*_{N+1} = 0$,
$y^*_k = \text{argmin}_{y_k}\{G_k(y_k)\}\ \forall\, k = 1, 2, ..., N$.
(ii) $R_k(\tilde{x}_k)$ *is concave in* x_k, q_{k-1} *and* q_{k-2} *and* $G_k(y_k)$ *is concave in* $y_k\ \forall\, k = 1, 2, ..., N$.
(iii) Optimal ordering policy of the retailer is a base stock policy.
(iv) The optimal order-up-to level at the last period is:

$$y^*_N = F^{-1}\left(\frac{b_N + c_N - p_N + \gamma(p_{N+1} - c_{N+1})}{-p_N - \gamma\alpha\Delta c_{N+1} + \gamma(p_{N+1} - c_{N+1}) - h_N - b_N}\right).$$

□

In Theorems 3 and 4, we prove that order-up-to type policy is optimal. Also, we observe that the order-up-to levels of the models with single protection period and double protection period is the same, similar to SBM. This can be the consequence of allowing backorders. The same reasoning can be applied here. We observe that the concavity of the profit to go function is preserved in an intermediate period if the total of the holding, backorder costs and the selling price of the item is higher than the discounted total of the protection credit that will be delivered and the selling price of the item for the next period (i.e. $(-\gamma\alpha\Delta c_{k+1} + \gamma p_{k+1} - p_k - h_k - b_k) \leq 0$). Also, at the end of the horizon, the concavity is preserved if the total holding, backorder costs and selling price of the item is higher than the discounted total of the protection credit that will be paid to the retailer at the end of the horizon and the profit margin of the retailer that is obtained from selling a single item to a customer (i.e. $(-p_N - \gamma\alpha\Delta c_{N+1} + \gamma(p_{N+1} - c_{N+1}) - h_N - b_N) \leq 0$). Otherwise, transactions between the retailer and the customers would be meaningless.

MBM is applicable for the products whose demand and manufacturing rate is high. The customers has a lot of choices to buy the product. Therefore, they are not willing to pay the reservation cost i.e. to be charged with the former price of the item. In these cases, using MBM for modeling the environment is more adequate. Most of the computer products can be considered in this set.

3.3 Lost Sales Model

Unlike the SBM and MBM, in LSM the unsatisfied demand is lost. The retailer incurs a shortage cost and loses profit of each item that cannot be satisfied.

In LSM, there are some changes in the model construction and the evolution equation. Similar to the MBM, we use a maximization objective. The state vector is given by

$$\tilde{x}_k = (x_k,\ q_{k-a},\ q_{k-a+1}, ..., q_{k-l},\ q_{k-l+1}, ..., q_{k-1})\,.$$

The evolved state vector for the $(k+1)^{st}$ period is:

$$\tilde{x}_{k+1} = (\max(0, x_k + q_{k-l} - w_k),\ q_{k-a+1},\ q_{k-a+2}, ..., q_{k-l},\ q_{k-l+1}, ..., q_{k-1},\ q_k)\,.$$

The reason for this change is the fact that the retailer's net inventory level at the beginning of any period cannot be less than zero due to lost sales. The profit to go function of the retailer at the k^{th} period when the system state is $\tilde{x}_k$ is given by the following:
LSM-DP:

$$\begin{aligned}
R_{N+l+1}(\tilde{x}_{N+1}) &= -\alpha\Delta c_{N+1}\min\{q_{N+l+1-a} + ...q_N, x_{N+l+1}\}^+ \\
R_k(\tilde{x}_k) &= \max_{q_k=0}\{-c_k q_k + p_k E_{W_k}\left(\min(\max(0, x_k + q_{k-l}), W_k)\right) \\
&\quad - L(x_k + q_{k-l}) - \alpha(c_k - c_{k-1})\min\{q_{k-a} + ... + q_{k-l} + v_k, x_k + v_k\}^+ \\
&\quad + \gamma E_{W_k}(R_{k+1}(\tilde{x}_{k+1}))\}\ \ \forall\ \ k = N+1, ..., N+l, \\
R_k(\tilde{x}_k) &= \max_{q_k\geq 0}\{-c_k q_k + p_k E_{W_k}\left(\min(\max(0, x_k + q_{k-l}), W_k)\right) \\
&\quad - L(x_k + q_{k-l}) - \alpha(c_k - c_{k-1})\min\{q_{k-a} + ... + q_{k-l} + v_k, x_k + v_k\}^+ \\
&\quad + \gamma E_{W_k}(R_{k+1}(\tilde{x}_{k+1}))\}\ \ \forall\ \ k = 1, 2, ..., N,
\end{aligned}$$

where

$$\begin{aligned}
L(y) &= h_k\int_0^y (y - w_k)dF(w_k) + b_k\int_y^\infty (w_k - y)dF(w_k), \\
&\quad F_{W_k}(0) = 1\ \ \forall\, k = 1, ..., l.
\end{aligned}$$

The following DP formulations are valid for the zero lead time where the protection age limits are determined as 1 and 2 by the supplier, respectively:

LSM-DP-1:

$$
\begin{aligned}
R_{N+1}(\tilde{x}_{N+1}) &= -\alpha\Delta c_{N+1}\min(x_{N+1}, q_N)^+ \\
R_k(\tilde{x}_k) &= \max_{q_k\geq 0}\{-c_k q_k + p_k E_{W_k}\left(\min(x_k+q_k,0)\right) \\
&\quad - L(x_k+q_k) - \alpha\Delta c_k \min(x_k, q_{k-1})^+ \\
&\quad + \gamma E_{W_k}(R_{k+1}(\tilde{x}_{k+1}))\}\ \ \forall\, k = 1,2,...,N,
\end{aligned}
$$

LSM-DP-2:

$$
\begin{aligned}
R_{N+1}(\tilde{x}_{N+1}) &= -\alpha\Delta c_{N+1}\min(x_{N+1}, q_N + q_{N-1})^+ \\
R_k(\tilde{x}_k) &= \max_{q_k\geq 0}\{-c_k q_k + p_k E_{W_k}\left(\min(x_k+q_k,0)\right) \\
&\quad - L(x_k+q_k) - \alpha\Delta c_k \min(x_k, q_{k-1}+q_{k-2})^+ \\
&\quad + \gamma E_{W_k}(R_{k+1}(\tilde{x}_{k+1}))\}\ \ \forall\ \ k = 1,2,...,N.
\end{aligned}
$$

In the following theorem, we characterize the ordering decision of the last period for $a = 1$, $a = 2$ and $l = 0$.

Theorem 5 *For LSM-DP-1 and LSM-DP-2, optimal order-up-to level at the last period is given by*

$$
y_N^* = F^{-1}\left(\frac{p_N - c_N - b_N}{p_N - \alpha\Delta c_{N+1} + h_N + b_N}\right).
$$

Chapter 4

Numerical Study

The objective in our numerical study is to analyze the impact of the price protection policies on the optimal replenishment behavior of the retailer and different performance metrics of the retailer and supplier. We select the following policy parameters and the performance metrics for our analysis.

1. Order-up-to levels: This is the optimal decision rule for a rational profit maximizer retailer.

2. Expected profit of the retailer: Under the assumption of the existence of rational retailer, this parameter is the outcome of the transaction.

3. Expected cost for the protected items: This cost is incurred by the supplier and it is the result of the protection policies.

4. Expected revenue of the supplier: The revenue that the supplier generates from the retailer.

5. Supplier's profit: The supplier earns revenue from the products that the retailer is ordering from the supplier and reimburses the cost difference of the protected inventory if there is an acquisition cost decline. Supplier's profit is the difference between the expected revenue of the supplier and the expected cost for the protected inventory of the retailer. The operational,

manufacturing, holding, backorder and opportunity costs of the supplier are ignored while computing the supplier's profit.

6. Type 1 service level: Probability of no stock out during the horizon.

7. Type 2 service level: Fraction of demand satisfied directly from the shelf.

In the first part of our study, we analyze the Modified Backorder Model (MBM) and in the second part we analyze the Lost Sales Model (LSM). We verify that the optimal ordering policy for the retailer in MBM is order-up-to as it is also shown in Theorems 3 and 4. Also we observe numerically that order-up-to policy is optimal in LSM for the retailer.

For both parts of the study, we are solving a 6-period problem with zero lead time and an ample supplier that offers the protection age limit to the retailer as one or two. We assume that the demand is stationary during the horizon. We explore the performance metrics' responses under two different distributions that are Poisson and negative binomial (NBD). We study Poisson distribution under different means and NBD under different parameters that lead to the same mean but different variances. We assume that the discount factor is one throughout the horizon. Furthermore, we consider the problem parameters listed in Table 4.1 and the cost decline structures are listed in Table 4.2.

Table 4.1: Problem Parameters

Parameter	Type
Cost Decline (CD)	Linear, Increasing, Decreasing
Price(P)	$c_i + 15$, $c_i + 30$, $1.15c_i$, $1.30c_i$
Backorder Cost(B)	$0.10c_1$, $0.30c_1$
Holding Cost	$0.10c_i$
Protection Age Limit(a)	0, 1, 2
Protection Credit Limit(α)	0.8, 1
Discount Factor (γ)	1
Model	MBM, LSM
Demand	Poisson($\lambda = 4$), Poisson($\lambda = 5$), Poisson($\lambda = 7$) NBD(2,0.4), NBD(3,0.6), NBD(4,0.8)

Table 4.2: Cost Decline Structures

Cost Decline Type (CD)	Acquisition Cost of Each Period						
	c_1	c_2	c_3	c_4	c_5	c_6	c_7
Linear Decline (LD)	100	90	80	70	60	50	45
Increasing Decline (ID)	100	98	94	86	70	50	45
Decreasing Decline (DD)	100	80	64	56	52	50	45

The order-up-to levels and the expected profit of the retailer are derived by solving the dynamic programming formulation of the problem from the last period to the first one. At the end of the horizon, if the net inventory level at the retailer is greater than zero, the protection credit is calculated and paid to the retailer by the supplier in both MBM and LSM. If the retailer is operating with the MBM, the excess demand at the last period is assured to be satisfied by the retailer at the end of the horizon. If the retailer is operating with LSM, the excess demand at the last period is simply lost. The calculation of the expected profit of the retailer, expected cost for the protected items, expected revenue of the supplier, supplier's profit, Type 1 and Type 2 service levels are done after determining the optimal order-up-to levels and feeding them into the simulation program that is coded in C. Each problem instance is run for 1500 replications. After completing the replications, the average of the related costs are determined and this process is repeated for all the problem instances we tabulated in Table 4.1. We choose the simulation in order to calculate the performance metrics since it is less complicated especially while computing the service levels. We double check the expected profit of the retailer with the result we obtained in the dynamic programming problem.

Price protection policy is costly from the supplier's perspective since the supplier promises the reimbursement of the unsold inventory in the retailer's stocks in case acquisition cost of the item declines. New component in the supplier's costs is the protection cost. Cost component reflects the cost of increasing the availability of the product in the market (it is the cost of increasing flow rate of the newer products in the supply channel). The protection cost for the supplier obviously affects the supplier's profit. It is intuitive that increasing protection age limit or the credit, α, results higher protection costs and higher supplier's revenue, however we cannot say much about the supplier's profit as we do not consider ever cost component of the supplier here.

4.1 Modified Backorder Model

In all instances we observe that the profit of the retailer increases as the supplier increases the protection age limit or the credit. Table 4.3 shows the expected retailer profit when there is no price protection, and the percentage improvements on the expected retailer profit under different price protection terms when compared with the base case. We observe some instances where the retailer can increase his profit by 214.74% after the supplier introduces a protection policy to the chain. Therefore, the introduction of price protection policies make the retailer's position better in comparison to the base case.

The minimum profit is attained by the retailer at all cost decline structures when the retailer is operating with higher backorder cost ($0.30c_1$) and small profit margin ($1.15c_i$). Therefore, even a small increase in the retailer's profit has a significant effect as a percent increase. As a result, maximum percent increase in the retailer's profit is achieved when the supplier introduces protection policies to the channel in these cases.

Table 4.3: Percent Increase in Retailer Profit Under Different Price Protection Policies (Poisson($\lambda = 5$))

Problem			Retailer	$\alpha = 0.8$		$\alpha = 1$	
CD	P	B	Profit (a=0)	$a = 1$ %inc	$a = 2$ %inc	$a = 1$ %inc	$a = 2$ %inc
LD	c_i+15	$0.10c_1$	307.62	7.22	7.72	10.01	10.62
	c_i+30		757.62	2.93	3.13	4.06	4.31
	$1.15c_i$		182.72	14.48	15.30	19.17	20.25
	$1.30c_i$		507.82	6.03	6.33	7.72	8.27
	c_i+15	$0.30c_1$	175.01	32.34	34.68	41.21	45.71
	c_i+30		625.01	9.05	9.71	11.53	12.79
	$1.15c_i$		56.99	100.67	108.62	129.38	144.31
	$1.30c_i$		389.51	14.78	16.18	19.27	21.58
ID	c_i+15	$0.10c_1$	305.83	6.41	6.97	9.17	9.89
	c_i+30		755.83	2.59	2.81	3.70	4.00
	$1.15c_i$		217.10	10.86	11.68	14.78	15.80
	$1.30c_i$		578.55	4.74	5.05	6.28	6.79
	c_i+15	$0.30c_1$	175.98	27.15	29.70	35.22	39.02
	c_i+30		625.98	7.62	8.34	9.90	10.96
	$1.15c_i$		92.64	53.82	58.69	69.83	77.93
	$1.30c_i$		459.29	11.30	12.28	14.70	16.52
DD	c_i+15	$0.10c_1$	313.74	6.89	7.15	9.42	9.93
	c_i+30		763.74	2.82	2.93	3.86	4.08
	$1.15c_i$		152.77	16.35	17.19	22.39	23.61
	$1.30c_i$		442.98	6.39	6.68	8.62	9.16
	c_i+15	$0.30c_1$	191.45	27.09	29.24	35.73	38.94
	c_i+30		641.45	8.08	8.72	10.66	11.62
	$1.15c_i$		36.23	150.08	161.46	197.27	214.74
	$1.30c_i$		331.12	17.14	18.39	22.48	24.60

Table 4.4 shows that the expected profit of the supplier when there is no price protection, expected protection cost and percentage improvements on the supplier's profit under different price protection terms. The expected profit of the supplier is calculated by taking the difference between the expected revenue of the supplier and expected cost for the protected items.

$$Expected\ Profit\ of\ the\ Supplier = Expected\ Revenue\ of\ the\ Supplier - Expected\ Cost\ for\ the\ Protected\ Items.$$

We observe that the expected protection cost is non-decreasing in the protection age limit and credit. However, we cannot observe such a monotonicity for the supplier's profit. Since it strictly depends on the problem parameters and order-up-to levels. Also, in most of the cases the supplier is hurt from protection policies. However, there are cases where win-win situation is observed. Therefore, if price protection policies are managed well, both of the players in the supply chain can be better off.

Table 4.4: Protection Cost and Percent Increase in Supplier's Profit Under Different Price Protection Policies (Poisson($\lambda = 5$))

Problem			Protection Cost	Supplier's Profit	Protection Cost				Supplier's Profit % Increase			
CD	P	B			$\alpha = 0.8$		$\alpha = 1$		$\alpha = 0.8$		$\alpha = 1$	
			$a = 0$	$a = 0$	$a = 1$	$a = 2$	$a = 1$	$a = 2$	$a = 1$	$a = 2$	$a = 1$	$a = 2$
LD	c_i+15	$0.10c_1$	0	2177.33	34.21	35.55	43.40	46.17	2.09	1.67	1.09	0.91
	c_i+30		0	2197.52	34.09	35.55	42.95	44.58	0.40	1.15	0.37	0.34
	$1.15c_i$		0	2218.63	34.03	35.61	42.98	51.15	0.33	-0.36	-0.56	-0.61
	$1.30c_i$		0	2194.57	34.54	36.67	48.53	63.32	0.72	0.21	0.37	1.58
	c_i+15	$0.30c_1$	0	2299.87	57.63	68.75	87.49	107.68	-1.69	-2.02	-3.04	-2.47
	c_i+30		0	2295.64	57.05	67.90	85.60	107.24	-1.21	-1.53	-1.82	-1.84
	$1.15c_i$		0	2323.16	57.67	73.33	93.16	105.55	-2.85	-2.27	-.358	-2.30
	$1.30c_i$		0	2311.00	57.09	79.44	92.64	113.65	-1.65	-1.60	-3.04	-1.79
ID	c_i+15	$0.10c_1$	0	2415.81	28.73	35.33	42.10	46.17	1.52	1.54	1.08	0.90
	c_i+30		0	2425.38	28.32	35.49	42.51	44.54	0.41	1.61	1.71	0.68
	$1.15c_i$		0	2463.91	34.18	35.77	42.40	44.27	0.12	-0.55	-0.73	-0.89
	$1.30c_i$		0	2440.08	34.17	37.09	54.94	66.52	0.41	-0.11	0.31	1.38
	c_i+15	$0.30c_1$	0	2523.92	57.21	62.55	72.96	103.67	-0.92	-1.54	-2.62	-1.72
	c_i+30		0	2503.95	56.45	61.41	74.12	106.17	0.20	-0.42	-2.05	-0.24
	$1.15c_i$		0	2539.62	56.33	60.41	86.76	109.08	-1.65	-1.59	-2.61	-1.05
	$1.30c_i$		0	2513.74	56.92	61.40	86.59	110.44	0.05	-0.58	-1.57	-0.19
DD	c_i+15	$0.10c_1$	0	1927.50	27.92	29.07	47.04	48.78	2.14	1.74	1.79	1.60
	c_i+30		0	1933.49	28.00	28.70	44.26	47.90	1.03	1.80	2.56	1.46
	$1.15c_i$		0	1982.23	35.49	37.49	50.49	53.06	0.25	-0.45	-0.53	-0.85
	$1.30c_i$		0	1969.68	37.00	38.49	60.22	62.31	0.16	-0.35	0.03	0.74
	c_i+15	$0.30c_1$	0	2047.97	65.56	71.05	84.30	100.75	-0.81	-1.57	-2.82	-2.22
	c_i+30		0	2033.11	65.46	70.59	84.81	97.46	0.15	-0.54	-2.30	-0.85
	$1.15c_i$		0	2058.67	66.69	69.94	95.40	113.07	-1.63	-1.49	-2.81	-1.39
	$1.30c_i$		0	2044.47	64.97	69.32	94.78	112.42	-0.19	-0.85	-2.12	-0.78

Table 4.5: Percent Increase in Retailer's Service Levels Under Different Price Protection Policies (Poisson($\lambda = 5$))

Problem			Retailer's Type 1 Service Level	Type 1 Service Level				Retailer's Type 2 Service Level	Type 2 Service Level			
				$\alpha = 0.8$		$\alpha = 1$			$\alpha = 0.8$		$\alpha = 1$	
CD	P	B	$a = 0$	$a = 1$ %inc	$a = 2$ %inc	$a = 1$ %inc	$a = 2$ %inc	$a = 0$	$a = 1$ %inc	$a = 2$ %inc	$a = 1$ %inc	$a = 2$ %inc
LD	c_i+15	$0.10c_1$	0.42	32.90	33.30	34.00	34.30	0.71	12.00	12.27	12.24	12.34
	c_i+30		0.41	35.26	35.56	35.26	35.56	0.70	12.97	13.07	12.97	13.07
	$1.15c_i$		0.40	37.57	40.58	37.57	45.66	0.70	13.71	13.96	13.71	15.36
	$1.30c_i$		0.41	35.54	36.99	42.12	51.68	0.70	12.91	13.13	15.03	17.93
	c_i+15	$0.30c_1$	0.66	6.53	10.14	14.32	18.19	0.86	2.69	3.93	5.10	6.73
	c_i+30		0.66	6.39	10.28	12.94	18.03	0.86	2.46	4.11	4.80	6.50
	$1.15c_i$		0.68	5.09	9.64	13.48	14.34	0.87	1.68	3.47	4.64	5.08
	$1.30c_i$		0.71	0.12	7.73	8.25	12.03	0.88	0.05	2.82	2.96	4.42
ID	c_i+15	$0.10c_1$	0.45	17.41	23.85	24.85	24.85	0.73	6.75	9.42	9.18	9.18
	c_i+30		0.45	18.78	24.26	23.54	25.52	0.73	7.73	9.23	9.29	9.73
	$1.15c_i$		0.43	28.80	30.93	28.80	30.93	0.72	10.48	10.95	10.53	10.97
	$1.30c_i$		0.47	18.06	19.33	23.75	27.05	0.74	7.29	7.50	9.55	10.42
	c_i+15	$0.30c_1$	0.66	6.91	7.36	9.20	13.62	0.86	2.67	2.74	3.22	5.08
	c_i+30		0.67	5.81	6.66	8.16	12.39	0.86	2.38	2.92	2.97	4.89
	$1.15c_i$		0.65	8.42	8.55	12.65	15.63	0.86	2.99	3.04	4.44	5.64
	$1.30c_i$		0.67	5.32	6.85	9.87	14.00	0.86	2.20	2.55	3.73	5.14
DD	c_i+15	$0.10c_1$	0.45	18.27	18.50	30.39	30.39	0.72	7.68	7.77	11.71	11.71
	c_i+30		0.44	19.27	19.51	30.73	32.48	0.72	7.77	7.86	11.66	12.11
	$1.15c_i$		0.46	26.75	27.87	30.99	32.32	0.74	9.27	9.76	10.96	11.11
	$1.30c_i$		0.50	17.27	18.05	27.12	27.17	0.76	6.24	6.39	9.61	9.67
	c_i+15	$0.30c_1$	0.68	11.46	11.84	13.51	14.71	0.87	4.45	4.55	4.78	5.58
	c_i+30		0.69	10.61	11.20	11.59	14.08	0.87	3.93	4.36	4.30	5.30
	$1.15c_i$		0.68	13.24	13.28	16.42	17.67	0.86	4.85	4.87	5.86	6.34
	$1.30c_i$		0.70	8.68	9.48	12.43	13.85	0.88	3.26	3.46	4.42	5.08

It is obvious that the price protection policies increase the service levels of the retailer since it motivates the retailer to order more thus leads to less backorders or stock out occasions. In Table 4.5, we observe that both Type 1 and Type 2 service levels are non-decreasing in the protection age limit and credit. In all cost decline structures, percent increase in the service levels are more significant if the retailer is operating with a lower backorder cost.

In all instances, we observe that order-up-to policy is optimal and order-up-to levels are non-decreasing in the protection age limit and credit offered by the supplier. Also, for a given protection credit, α, we observe that the expected profit of the retailer increases as the supplier increases the protection age limit. In Table 4.6, the results are given for different cost decline structures. In DD we observe a non-monotonicity in the order-up-to levels of the retailer, unlike the other cost structures. In DD, the price is so high at the beginning of the horizon and declines very rapidly. Therefore, the retailer tends to buy later in time.

Table 4.6: Retailer Profit and Order-up-to Levels Under Different Cost Decline Structures (Poisson($\lambda = 5$),$b_i = 0.10c_1$, $h_i = 0.10c_i$, $p_i = c_i + 30$)

Decline Type	Protection Policy(a)	Protection Credit(α)	Order-up -to Levels						Retailer Profit
			S_1	S_2	S_3	S_4	S_5	S_6	
LD	0	-	4	4	4	4	4	3	757.63
	1	1	5	5	5	5	5	3	788.38
	2	1	5	5	5	5	5	3	790.27
	1	0.8	5	5	5	5	5	3	779.80
	2	0.8	5	5	5	5	5	3	781.32
ID	0	-	5	4	4	4	4	3	755.83
	1	1	5	5	5	5	5	3	783.82
	2	1	5	5	5	5	5	3	786.04
	1	0.8	5	5	5	4	5	3	775.39
	2	0.8	5	5	5	5	5	3	777.09
DD	0	-	3	4	4	5	5	3	763.74
	1	1	5	5	5	6	5	3	793.27
	2	1	5	5	5	6	5	3	794.87
	1	0.8	4	5	5	5	5	3	785.30
	2	0.8	4	5	5	5	5	3	786.12

We now analyze a specific example in details, where we study the impact of the price protection policies to the order-up-to levels and service levels of the retailer and the supplier's protection cost, revenue and profit. In this example we choose the demand distribution as Poisson with mean 5 and LD is assumed for cost decline structure. The supplier is offering a full protection to the retailer ($\alpha = 1$) and the protection age limit is assumed to be 1 or 2. Selling price is determined by adding a fixed mark-up (+30) to the acquisition cost. The responses of the performance metrics is explored first of all when the backorder cost is 10% of the highest acquisition cost and then when the backorder cost is 30% of the highest acquisition cost.

In the presence of price protection, the retailer is encouraged to order more, thus the order-up-to levels are non-decreasing and this is followed by less backorder costs and more expected profit. Also, at each price decline the revenue of the retailer is incremented by a certain amount depending on the number of the protected items, protection credit and the decline in the acquisition cost. When we analyze the expected profit of the retailer under different protection age limits in Table 4.6, we see that the retailer profit increases very rapidly with the introduction of the protection mechanism. Later on, the impact of increasing the protection age limit decreases. Shifting from no protection policy to the 1-period protection policy increases the profit much more than shifting from 1-period protection mechanism to the 2-period protection mechanism with the same protection credits.

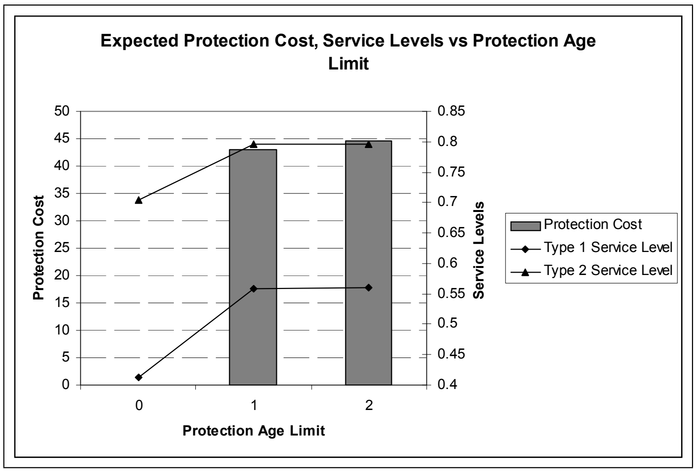

Figure 4.1: Expected Protection Cost for Supplier, Type 1 and Type 2 Service Levels of the Retailer vs Protection Age Limit (LD, Poisson($\lambda = 5$), $\alpha = 1$, $b_i = 0.10c_1$, $p_i = c_i + 30$)

In Figures 4.1 and 4.2, the impact of price protection policy on the retailer and supplier performance are displayed for the specific example when the backorder cost is 10% of the highest acquisition cost. Introducing a protection mechanism to the system increases the service levels of the retailer and increases the protection cost and expected revenue of the supplier. Since the order-up-to levels increase, the retailer orders more and faces less stock out occasions and incurs less back-orders. Thus the service levels of the retailer increase. Also, expected revenue of the supplier increases after applying the protection mechanism to the system since the retailer orders more. However, increasing the protection age limit from 1 to 2 does not change the expected revenue of the supplier, because order-up-to levels do not change. As expected, as the age limit of the protection policy is increased, expected protection cost increases. However, we cannot say anything about the supplier's expected profit, since it strictly depends on the problem parameters. In Figure 4.2, we see that the expected profit of the supplier increases either the protection age limit is 1 or 2 when compared with the no protection case. However the increment is not so significant, it increase by 0.375% in the 1-period protection policy and it increases by 0.344% in the 2-period protection case when compared to the base case where there is no protection policy offered by the supplier. The highest supplier profit is achieved when he introduces the

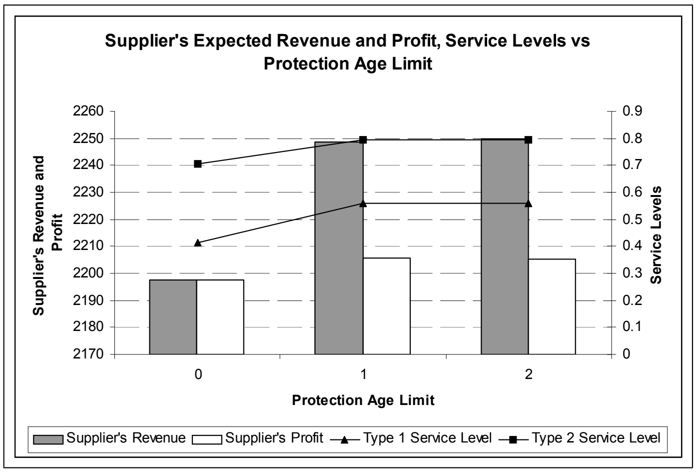

Figure 4.2: Expected Profit and Revenue for Supplier, Type 1 and Type 2 Service Levels vs Protection Age Limit(LD, Poisson($\lambda = 5$), $\alpha = 1$, $b_i = 0.10c_1$, $p_i = c_i + 30$)

age limit as 1. However, the highest service levels are attained when the supplier offers the retailer 2-period protection policy.

If we set the $b = 0.30c_1$ and analyze the performance metrics, we observe that the service levels, protection cost and expected revenue of the supplier increase as the age limit of the protection is increased.

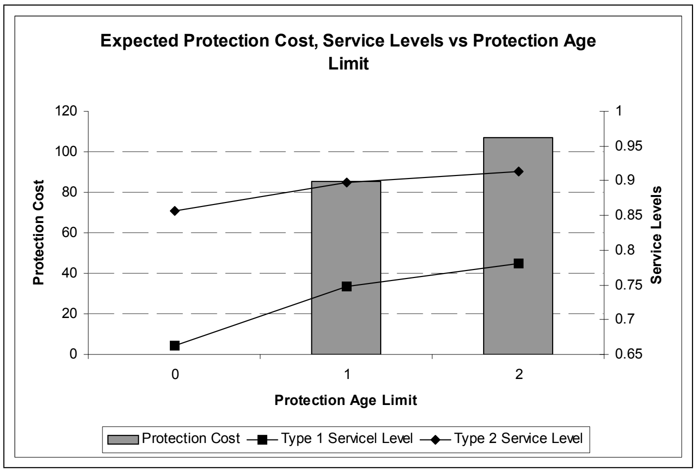

Figure 4.3: Expected Protection Cost for Supplier, Type 1 and Type 2 Service Levels of the Retailer vs Protection Age Limit (LD, Poisson($\lambda = 5$), $\alpha = 1$, $b_i = 0.30c_1$, $p_i = c_i + 30$)

In Figure 4.3, we observe that the increment in the protection cost when shifting from no price protection to 1-period price protection is much more than the increment when shifting from 1 to 2-period price protection policy. Also, the increase in the service levels in shifting from no protection to 1-period protection policy is much more than the increase in the service levels while shifting from 1 to 2-period protection policy. Although the protection cost increases as the protection age limit is increased, the increment is not so significant when the age limit is increased from 1 to 2 in comparison with the increment when the age limit is increased from 0 to 1 by the supplier. In Figure 4.4, we observe that the expected revenue of the supplier increases as the age limit is increased, similar to the protection cost situation, the increment is much more when the supplier increases the protection age limit from 0 to 1 than the case where he increases it from 1 to 2. When we analyze the expected profit of the supplier, we see that the maximum profit is attained when the supplier does not offer any protection policy to the retailer. However, this case provides the worst customer service level. In this situation, increasing the service levels of the retailer hurts the supplier very badly.

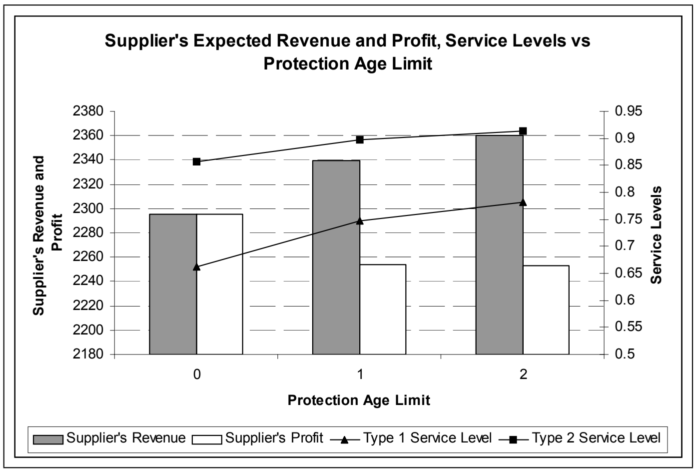

Figure 4.4: Expected Revenue and Profit for Supplier, Type 1 and Type 2 Service Levels vs Protection Age Limit (LD, Poisson($\lambda = 5$), $\alpha = 1$, $b_i = 0.30c_1$, $p_i = c_i + 30$)

We finally investigate the impact of the protection credit and demand variability in MBM. First we examine the protection credit. As we increase the protection credit, expected profit of the retailer increases. We observe in Figure 4.5, the expected profit of the retailer is higher if he operates with 2-period protection policy with the same protection credits. Also, in Figure 4.5 we observe that the difference in the Retailer's profit with 1 period protection and 2 period protection increases as the protection credit is increased.

In order to see the effect of the variance to the price protection policy, we use the NBD with the same mean and different variances. In Table 4.7, we do not observe a monotonic relation between the variance and the % increase in the retailer profit.

4.2 Lost Sales Model

Table 4.8 shows the expected retailer profit when there is no protection and the percentage profit improvements under different price protection terms in LSM.

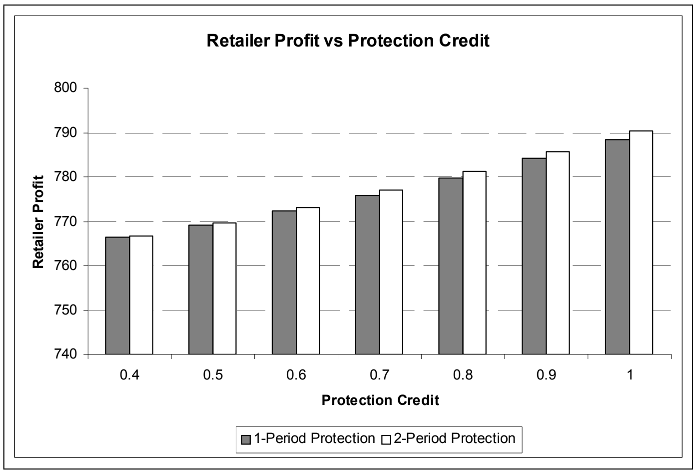

Figure 4.5: Expected Profit of the Retailer vs Protection Credit (LD, Poisson($\lambda = 5$), $b_i = 0.10c_1$, $p_i = c_i + 30$)

Table 4.7: Retailer Profit Under Different Distributions (LD, $b_i = 0.10c_1$, $p_i = c_i + 30$)

Distribution Type	Mean	Variance	Retailer Profit	% Increase in Retailer Profit			
				$\alpha = 0.8$		$\alpha = 1$	
				$a = 1$	$a = 2$	$a = 1$	$a = 2$
NBD(2, 0.4)	5	7.5	748.65	2.69	2.72	3.69	3.78
NBD(3, 0.6)	5	3.33	795.71	1.54	1.54	2.37	2.37
NBD(4, 0.8)	5	1.25	816.03	2.20	2.20	2.75	2.75

We observe that the retailer can increase his profit up to 6704.49% under certain problem parameters. Therefore, under price protection strategy the retailer gets better in comparison with the base case where no protection policies are implemented. The highest percent increase is observed in the same case with the MBM. The same reasoning can also be applied here. Another observation is the fact that the percent increase achieved in LSM is higher than the percent increase achieved in MBM, since operating with LSM is more costly so the retailer gets much better off when compared with the MBM by the tiniest improvements in his profit.

Table 4.8: Percent Increase in Retailer Profit Under Different Price Protection Policies (Poisson($\lambda = 5$))

Problem			Retailer	$\alpha = 0.8$		$\alpha = 1$	
CD	P	B	Profit (a=0)	$a = 1$ %inc	$a = 2$ %inc	$a = 1$ %inc	$a = 2$ %inc
LD	c_i+15	$0.10c_1$	201.31	24.66	26.57	31.82	34.49
	c_i+30		585.17	10.64	11.95	14.15	15.93
	$1.15c_i$		114.36	37.30	40.67	49.69	53.90
	$1.30c_i$		396.15	14.54	15.90	18.84	21.42
	c_i+15	$0.30c_1$	116.71	59.40	66.53	77.66	86.57
	c_i+30		519.96	16.15	17.78	20.25	22.99
	$1.15c_i$		21.29	299.95	339.04	398.20	447.07
	$1.30c_i$		319.50	25.05	27.65	31.71	35.20
ID	c_i+15	$0.10c_1$	200.39	21.30	23.55	28.40	31.21
	c_i+30		579.81	9.80	11.09	13.01	15.11
	$1.15c_i$		143.58	26.47	29.61	36.21	40.14
	$1.30c_i$		160.86	11.18	12.52	14.40	16.81
	c_i+15	$0.30c_1$	111.25	56.44	64.85	74.91	85.84
	c_i+30		519.25	13.93	15.80	17.96	20.43
	$1.15c_i$		49.51	118.05	135.59	157.40	181.46
	$1.30c_i$		386.49	17.85	20.37	23.26	26.41
DD	c_i+15	$0.10c_1$	214.59	21.09	22.59	28.43	30.69
	c_i+30		600.69	10.26	11.09	13.44	14.83
	$1.15c_i$		91.63	44.74	48.03	60.44	64.62
	$1.30c_i$		343.28	16.12	17.21	21.14	23.43
	c_i+15	$0.30c_1$	132.91	50.87	54.79	66.48	73.22
	c_i+30		542.22	14.28	15.83	18.67	20.79
	$1.15c_i$		1.37	4658.42	5017.41	6096.82	6704.49
	$1.30c_i$		270.90	27.09	29.61	35.40	38.94

Table 4.9 shows the impact of price protection policies on the supplier's performance. The supplier's performance behaves similar to the MBM. Expected protection cost is non-decreasing in the protection age limit and credit. However, expected protection cost is higher since order-up-to levels in LSM is higher than the MBM. Also, in LSM supplier is observed to be hurt from introducing protection policies to the channel in several cases.

Table 4.9: Protection Cost and Percent Increase in Supplier's Profit Under Different Price Protection Policies (Poisson(λ = 5))

Problem			Protection	Supplier's	Protection Cost				Supplier's Profit % Increase			
CD	P	B	Cost	Profit	$\alpha = 0.8$		$\alpha = 1$		$\alpha = 0.8$		$\alpha = 1$	
			$a = 0$	$a = 0$	$a = 1$	$a = 2$	$a = 1$	$a = 2$	$a = 1$	$a = 2$	$a = 1$	$a = 2$
LD	c_i+15	$0.10c_1$	0	1888.89	57.09	61.04	72.64	85.59	7.23	6.97	5.79	6.43
	c_i+30		0	2079.98	78.21	91.55	106.45	116.70	1.41	2.46	3.67	3.26
	$1.15c_i$		0	1857.72	56.13	60.41	70.58	75.68	6.89	6.36	5.78	5.42
	$1.30c_i$		0	2078.93	57.56	87.18	98.46	14.26	-2.93	0.64	0.32	1.36
	c_i+15	$0.30c_1$	0	2130.93	85.03	94.39	108.02	117.24	2.22	1.39	0.40	0.45
	c_i+30		0	2202.76	84.97	100.44	114.43	151.26	-0.98	-0.98	-1.43	-0.38
	$1.15c_i$		0	2090.17	83.36	91.37	105.98	113.09	2.15	1.96	0.48	1.41
	$1.30c_i$		0	2129.38	84.80	93.00	107.65	132.62	2.76	2.19	0.73	2.35
ID	c_i+15	$0.10c_1$	0	2163.14	56.61	60.67	70.50	78.41	3.68	3.42	2.40	2.04
	c_i+30		0	2272.99	58.14	91.29	104.75	116.91	0.72	3.82	5.57	4.34
	$1.15c_i$		0	2088.07	55.64	60.37	69.42	75.00	5.50	5.00	4.53	4.19
	$1.30c_i$		0	2232.90	59.52	71.70	74.52	113.79	2.20	2.67	1.46	4.63
	c_i+15	$0.30c_1$	0	2401.37	75.99	94.41	106.43	116.09	-0.79	-0.59	-1.44	-1.41
	c_i+30		0	2409.42	82.93	93.38	107.55	149.87	0.09	-0.47	-2.15	0.03
	$1.15c_i$		0	2363.32	62.00	90.70	103.44	113.53	-2.16	-0.17	-1.46	-0.60
	$1.30c_i$		0	2427.43	83.65	93.53	105.29	116.44	-0.34	-0.94	-2.25	-1.64
DD	c_i+15	$0.10c_1$	0	1766.86	59.97	65.23	83.17	88.93	3.08	3.97	2.55	3.24
	c_i+30		0	1846.37	75.17	79.82	97.28	121.53	1.69	1.58	3.91	4.08
	$1.15c_i$		0	1711.09	56.55	60.55	72.01	79.71	3.35	2.83	2.13	2.54
	$1.30c_i$		0	1772.67	66.15	71.24	93.62	114.84	3.30	3.84	3.49	5.96
	c_i+15	$0.30c_1$	0	1894.60	78.39	85.02	113.32	128.32	1.86	1.09	1.26	1.86
	c_i+30		0	1954.10	95.50	103.71	120.52	140.74	1.57	0.99	-0.86	0.70
	$1.15c_i$		0	1854.52	74.88	79.76	109.06	116.32	1.25	1.17	0.88	2.36
	$1.30c_i$		0	1966.12	88.69	95.28	119.43	138.28	-0.02	-0.52	-1.68	-0.38

Table 4.10: Percent Increase in Retailer's Service Levels Under Different Price Protection Policies (Poisson($\lambda = 5$))

Problem			Retailer's Type 1 Service Level	Type 1 Service Level				Retailer's Type 2 Service Level	Type 2 Service Level			
				$\alpha = 0.8$		$\alpha = 1$			$\alpha = 0.8$		$\alpha = 1$	
CD	P	B	$a = 0$	$a = 1$ %inc	$a = 2$ %inc	$a = 1$ %inc	$a = 2$ %inc	$a = 0$	$a = 1$ %inc	$a = 2$ %inc	$a = 1$ %inc	$a = 2$ %inc
LD	c_i+15	$0.10c_1$	0.59	18.28	19.43	18.85	22.47	0.82	6.76	7.03	6.81	7.76
	c_i+30		0.71	11.32	11.83	16.12	13.12	0.88	4.04	4.19	6.18	6.19
	$1.15c_i$		0.55	23.55	24.00	23.54	24.01	0.79	8.25	8.57	8.26	9.66
	$1.30c_i$		0.71	0.12	9.43	9.80	11.65	0.88	0.05	3.35	3.37	4.08
	c_i+15	$0.30c_1$	0.74	11.38	11.56	12.84	12.84	0.90	4.00	4.13	4.23	4.26
	c_i+30		0.79	3.78	5.56	5.98	10.17	0.92	1.39	2.17	2.18	3.51
	$1.15c_i$		0.70	13.61	13.61	14.60	14.72	0.88	4.66	4.42	4.74	4.42
	$1.30c_i$		0.74	11.00	11.18	12.39	13.77	0.90	3.74	3.94	4.04	4.80
ID	c_i+15	$0.10c_1$	0.65	9.25	10.32	9.78	10.35	0.85	3.66	3.79	3.71	3.80
	c_i+30		0.71	6.63	12.40	16.48	17.10	0.88	2.59	4.63	6.47	6.69
	$1.15c_i$		0.57	19.53	20.85	19.53	20.85	0.80	7.52	7.58	7.52	7.70
	$1.30c_i$		0.68	10.21	12.23	10.59	17.23	0.86	3.57	4.60	3.72	6.17
	c_i+15	$0.30c_1$	0.77	4.00	6.38	7.60	7.66	0.91	1.46	2.37	2.56	2.60
	c_i+30		0.79	3.38	3.82	4.47	6.98	0.92	1.35	1.69	1.57	2.73
	$1.15c_i$		0.74	2.92	8.04	8.97	8.98	0.90	0.86	2.56	2.89	2.90
	$1.30c_i$		0.80	2.67	2.89	2.82	2.91	0.92	1.23	1.39	1.34	1.39
DD	c_i+15	$0.10c_1$	0.67	7.11	11.00	11.00	13.58	0.86	3.00	5.06	3.91	5.10
	c_i+30		0.73	6.60	6.59	11.19	14.27	0.89	3.12	3.12	4.64	5.72
	$1.15c_i$		0.60	12.93	14.35	12.95	16.61	0.82	4.52	4.83	4.52	5.54
	$1.30c_i$		0.66	12.41	15.67	15.10	19.41	0.86	4.18	5.46	5.22	6.72
	c_i+15	$0.30c_1$	0.75	9.06	9.15	12.50	13.74	0.90	3.36	3.47	4.43	4.96
	c_i+30		0.79	8.05	8.24	8.87	10.12	0.92	2.73	3.01	2.93	3.50
	$1.15c_i$		0.71	9.39	9.40	12.56	12.92	0.88	3.46	3.46	4.39	4.57
	$1.30c_i$		0.80	4.88	4.97	7.21	7.28	0.92	1.66	1.86	2.36	2.71

Table 4.10 shows the changes in the service levels due to different price protection policies under LSM. The results are similar to those in MBM. However in LSM, the retailer prefers to operate at higher service levels since the retailer incurs the extra opportunity cost of losing the demand in the market in the base case where no protection policy is implemented. Therefore, percent increase in the service levels is not so significant after introducing the protection policy to the channel when compared with the MBM.

In all of the instances we analyzed, we observe that the order-up-to policy is optimal. We also observe that the order-up-to levels and the expected profit of the retailer increase as the supplier increases the protection age limit. Table 4.11 shows the optimal order-up-to levels and corresponding expected retailer profit for a specific parameter set. We see that the order-up-to levels increase and the retailer earns more as the supplier offers more protection. Also, in DD we observe a non-monotonicity in the order-up-to levels of the retailer, different from the other decline structures. In DD, the price is so high at the beginning of the horizon and declines vary rapidly. Therefore, the retailer is prone to buy later in time.

In Table 4.11, we provide the order-up-to levels and the expected profit of the retailer if he operates with LSM. Similar to the MBM case, in the presence of the price protection policy the retailer is encouraged to order more, thus the order-up-to levels are increased and this increase is followed by less shortage costs, less demand loss and more expected profit. Also, as the protection age limit is increased by the supplier, more reimbursement will be gathered by the retailer in case a price decline in the acquisition cost. All these factors increase the expected profit of the retailer in the presence of the price protection policy in the system. When we analyze the expected profit of the retailer under different protection ages and with different price decline structures in Table 4.11, we see that the retailer profit increases very rapidly with the introduction of the protection mechanism. Later on the impact of increasing the protection period decreases. Shifting from no protection mechanism to the 1-period protection mechanism increases the profit more than shifting from 1-period protection mechanism to the 2-period protection mechanism with the same protection credits like MBM.

Table 4.11: Retailer Profit and Order-up-to Levels Under Different Cost Decline Structures ($N = 6$, Demand Poisson($\lambda = 5$), $LT = 0$,$b_i = 0.10c_1$, $h_i = 0.10c_i$, $p_i = c_i + 30$)

Decline Type	Protection Age Limit(a)	Protection Credit(α)	Order-Up-To Levels						Retailer Profit
			S_1	S_2	S_3	S_4	S_5	S_6	
LD	0	0	6	6	6	6	6	4	585.17
	1	1	7	7	7	7	7	5	667.99
	2	1	7	7	7	7	7	5	678.39
	1	0.8	6	7	7	7	7	4	647.44
	2	0.8	7	7	7	7	7	4	655.08
ID	0	0	7	6	6	6	5	4	579.81
	1	1	7	7	7	7	7	5	655.23
	2	1	7	7	7	7	7	5	667.40
	1	0.8	7	7	6	6	6	4	636.64
	2	0.8	7	7	7	7	7	4	644.12
DD	0	0	5	6	6	7	7	4	600.69
	1	1	6	7	7	8	7	5	681.44
	2	1	7	7	7	8	7	5	689.78
	1	0.8	6	7	7	7	7	4	662.34
	2	0.8	6	7	7	7	7	4	667.27

If we compare the order-up-to levels with the MBM, under the same operating parameters, we observe that although the order up to levels in the LSM is greater than the MBM model, the profit of the retailer in the LSM is less than that in the MBM model. This is because the market share of the retailer is preserved in the MBM model and all the demand is satisfied on time or with a delay. However, in the LSM, excess demand is lost and the retailer incurs shortage cost and loses profit for each demand that cannot be satisfied.

Now, we return to the example in which we investigate the impact of the price protection policy to the retailer's and supplier's performance. We consider the same case with the MBM. Figure 4.6 shows that the protection cost and the service levels of the retailer increase as the protection age limit is increased. Since the order-up-to levels increase with the introduction of the protection mechanism to the channel. However shifting from 1 to 2-period protection policy does not increase the protection cost as much as shifting from no protection to 1-period protection policy. Since the order-up-to levels do not change when the supplier

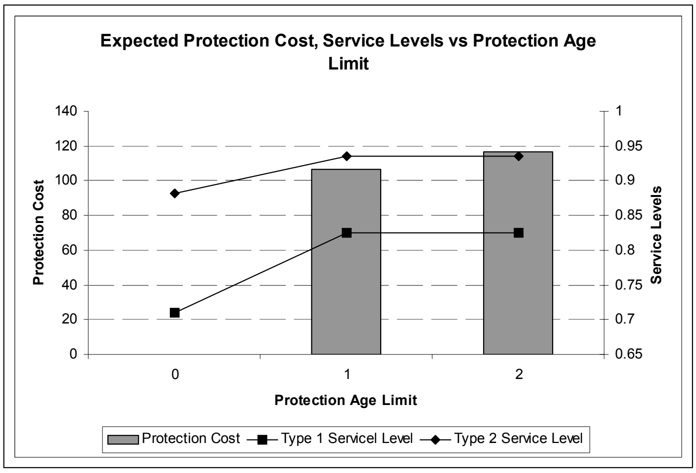

Figure 4.6: Expected Protection Cost for Supplier, Type 1 and Type 2 Service Levels of the Retailer vs Protection Age Limit (Linear Cost Decline, Demand Poisson($\lambda = 5$), $\alpha = 1$, $b_i = 0.10c_1$, $h_i = 0.10c_i$, $p_i = c_i + 30$)

offers 2-period protection policy instead of 1-period protection policy to the retailer. Also, the service levels increase as the supplier introduce a 1-period price protection policy to the channel, however the increase in the service levels are not so significant when he increase the protection age limit from 1 to 2.

In Figure 4.7, we observe that the expected revenue of the supplier increases as he offers 1-period protection policy to the channel, but it does not change if he increases the protection age limit from 1 to 2. This is because the order-up-to levels are the same when the age limit is 1 or 2. With these problem parameters, maximum profit is gathered by the supplier when the protection age limit is 1. Introducing a price protection policy whose age limit is 1 increases both the supplier's profit margin and the retailer's service levels. Therefore, both players of the channel gets better when the protection policy with an age limit 1 is applied. However shifting from 1-period protection policy to the 2-period protection policy decreases the supplier's profit. That is because the order-up-to levels do not change, the expected revenue of the supplier stays the same and the protection cost increases, thus the supplier's profit decreases. Similar observations are made for Type 1 and 2 service levels.

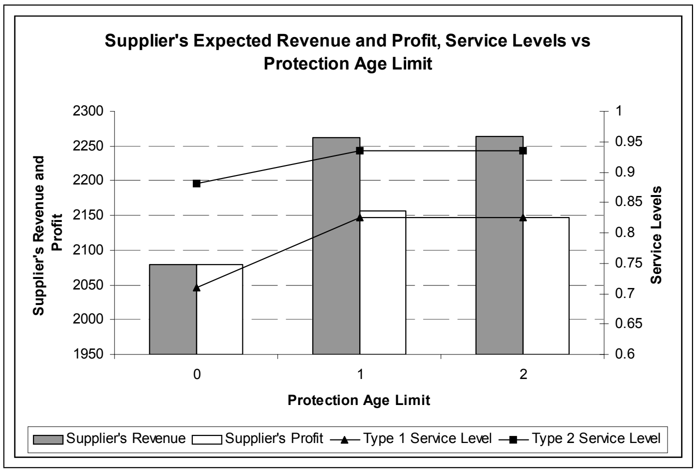

Figure 4.7: Expected Revenue and Profit for Supplier, Type 1 and Type 2 Service Levels of the Retailer vs Protection Age Limit (LD, Poisson($\lambda = 5$), $\alpha = 1$, $b_i = 0.10c_1$, $h_i = 0.10c_i$, $p_i = c_i + 30$)

If we set the shortage cost to the 30% of the largest acquisition cost, we obtain Figures 4.8 and 4.9 in which the protection cost, expected revenue of the supplier and the service levels of the retailer can be observed. Similarly, the protection cost and the service levels increase as the protection age limit is increased by the supplier. Since the order-up-to levels increase, the retailer loses less demand and incurs less shortage cost. Shifting from no protection policy to the 1-period protection policy increases the Type 1 and Type 2 service levels by 5.98% and 2.13 % respectively and increasing the protection age limit from 1 to 2 increases the the Type 1 and Type 2 service levels by 3.95% and 1.30% respectively.

In Figure 4.9, we observe that the expected revenue of the supplier increases as the supplier increases the protection age limit, since the order-up-to levels increase as the age limit is increased by the supplier. However, the highest profit is achieved by the supplier when he does not offer any protection policy to the channel. The least profit is obtained by the supplier when the supplier offers a 1-period protection policy to the channel. The expected profit of the supplier decreases by 1.43%, Type 1 and Type 2 service levels increase by 5.98% and 2.13%, respectively, when the supplier increases the protection age limit from 0 to 1. Also, the expected profit of the supplier decreases by 0.38%, type 1 and

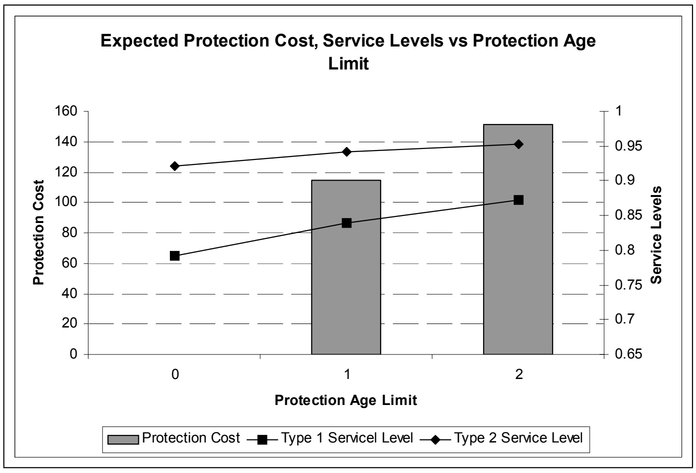

Figure 4.8: Expected Protection Cost for Supplier, Type 1 and Type 2 Service Levels of the Retailer vs Protection Age Limit (LD, Poisson($\lambda = 5$), $\alpha = 1$, $b_i = 0.30c_1$, $h_i = 0.10c_i$, $p_i = c_i + 30$)

type 2 service levels increase by 10.17% and 3.51%, respectively, when the supplier increases the protection age limit from 1 to 2. Therefore, in this case the supplier loses 0.38% on profits in return for a 10.17% and 3.51 % increases in the service levels of the retailer if he offers 2-period protection policy.

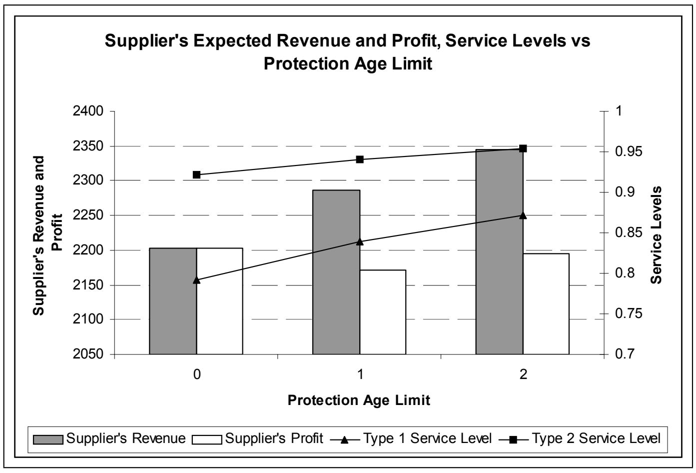

Figure 4.9: Expected Revenue and Profit for Supplier, Type 1 and Type 2 Service Levels of the Retailer vs Protection Age Limit (LD,Poisson($\lambda = 5$), $\alpha = 1$, $b_i = 0.30c_1$, $h_i = 0.10c_i$, $p_i = c_i + 30$)

Finally we explore the impact of the protection credit and the variance to the retailer's performance. As we increase the protection credit, expected profit for the retailer increases similar to the MBM. In Figure 4.10, we observe that the difference in the retailer's profit with 1 and 2-period protection increases as the protection credit is increased, since increasing the protection credit, α, has a more significant effect on the expected profit of the retailer when the protection age limit is 2.

In Table 4.12, we observe the impact of the variance to the retailer's profit under different price protection terms. If the variance is increased the percent increase in the retailer profit after implementing the price protection strategy increases. For this case, we can conclude that the retailer benefits more from the price protection policy when there is a higher variability in demand. Percent increases in LSM, are more than the percent increases in the MBM. Therefore, applying the price protection strategies in an LSM environment is more effective from the retailer's perspective.

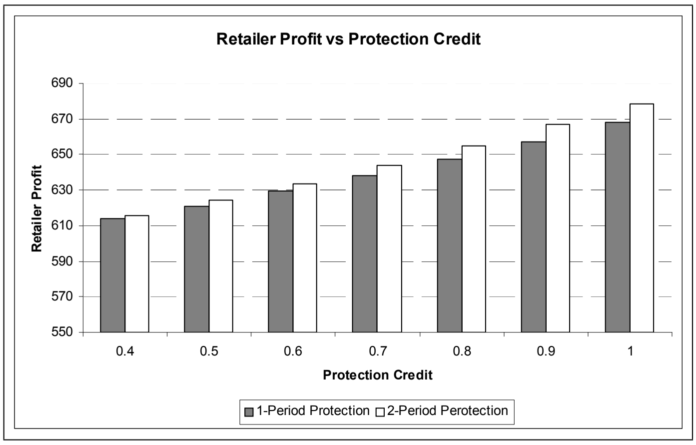

Figure 4.10: Expected Profit of the Retailer vs Protection Credit (LD, Poisson($\lambda = 5$), $b_i = 0.30c_1$, $p_i = c_i + 30$)

Table 4.12: Retailer Profit Under Different Distributions (LD, $b_i = 0.10c_1$, $p_i = c_i + 30$)

Distribution Type	Mean	Variance	Retailer Profit	% Increase in Retailer Profit			
				$\alpha = 0.8$		$\alpha = 1$	
				a=1	a=2	a=1	a=2
NBD(2, 0.4)	5	7.5	517.96	13.17	14.82	17.21	19.76
NBD(3, 0.6)	5	3.33	642.13	8.55	8.58	11.14	11.32
NBD(4, 0.8)	5	1.25	742.25	4.54	4.54	6.14	6.14

Chapter 5

Conclusion

In this thesis, we consider a single item, multi period, finite horizon inventory problem of a retailer who faces a stochastic demand and whose orders are fulfilled by an ample supplier. The selling price and the manufacturing costs of the product is decreasing in time, therefore the retailer is under risk of buying high and selling low. In order to motivate the retailer to order more, the supplier offers him a price protection policy whose age limit and credit are determined at the beginning of the planning horizon. In the study, the retailer's problem where the supplier offers a price protection policy is modeled in three different ways based on how the excess demand is treated. The first one is the Standard Backorder Model (SBM), the second one is the Modified Backorder Model (MBM), and the third one is the Lost Sales Model (LSM). It is proven that the base stock policy is optimal for the retailer if he is operating with SBM or MBM. It is shown that the order-up-to levels are non-decreasing and the retailer is better of as the supplier offers more protection in MBM and LSM. Increasing the order-up-to levels for the retailer results higher service levels in the retailer, the service levels of the retailer is verified to be non-decreasing in the protection age limit and the protection credit.

The impact of the price protection policy to the supplier is also explored. It is shown numerically that the expected protection cost and expected revenue are non-decreasing in the protection age limit and credit. However, we observe that the expected supplier's profit does not show a monotonic behavior as the protection credit or age limit increases. The relation strictly depends on the problem parameters. Although there exists cases where the profit of the both players increases with the price protection policy, in most of the cases the supplier may be hurt from the price protection policy.

Our study is the first study that investigates the outcomes of the price protection policy to the retailer in a stochastic demand and multi period environment. The analytical results that have been derived for the optimal ordering decision of the retailer are obtained for SBM and MBM for an age limit of 1 or 2. The impact of the price protection policy to the performance metrics of the supplier is numerically studied.

Future research can extend the analysis in many ways. First of all, optimal ordering policy of the retailer can be derived for a general protection age limit. A second direction could be to incorporate set-up costs for ordering. Also, we can study the circumstances under which price protection is pareto improving or coordinates the channel in a more detailed way under a multi period setting. In the model, prices could be the function of time and the protection age limit could be dynamic. The assumption that states $a \geq l$ could be relaxed and the response of the model can be explored.

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Appendix A

Proof of Theorems

Proof of Theorem 1: We need the following Lemmas in order to prove Theorem 1. Before stating the Lemmas, recall that the $F(w)$ is defined as the cdf of the non-negative r.v. W which denotes the stochastic demand and $E(W) = \overline{D}$ and $\Phi(y) = \int_0^y F(t)dt$.

Lemma 1 $\int_0^y tdF(t) = yF(y) - \Phi(y)$.

Proof: By integration by parts:

$$\begin{aligned}
\int_0^y tdF(t) &= \begin{pmatrix} u = t & du = dt \\ v = F(t) & dv = dF(t) \end{pmatrix} \\
&= tF(t) \mid_0^y - \int_0^y F(t)dt \\
&= yF(y) - \Phi(y)
\end{aligned}$$

□

Lemma 2 $\int_y^\infty (w - y)dF(w) = \overline{D} - y + \Phi(y)$.

$$\int_y^\infty (w-y)dF(w) = \int_y^\infty wdF(w) - y(1-F(y)) = \overline{D} - \int_0^y wdF(w) - y(1-F(y))$$

By using Lemma 1, we can write the above equation in the following way:

$$\int_y^\infty (w-y)dF(t) \;=\; \overline{D} - (yF(y) - \Phi(y)) - y + yF(y) = \overline{D} - y + \Phi(y).$$

□

Lemma 3 *For $y - x > 0$, $\int_0^y \min(y-w, y-x)dF(w) = \Phi(y) - \Phi(x)$.*

Proof:

$$\begin{aligned}\int_0^y \min(y-w, y-x)dF(w) &= \int_0^x (y-x)dF(w) + \int_x^y (y-w)dF(w) \\ &= yF(y) - xF(x) - \int_x^y wdF(w) \\ &= yF(y) - xF(x) - \left(\int_0^y wdF(w) - \int_0^x wdF(w)\right)\end{aligned}$$

By using Lemma 1,

$$\begin{aligned}\int_0^y \min(y-w, y-x)dF(w) &= yF(y) - xF(x) - (yF(y) - \Phi(y) - xF(x) + \Phi(x)) \\ &= \Phi(y) - \Phi(x).\end{aligned}$$

□

Lemma 4 $\int_0^\infty (y-w)dF(w) = y - \overline{D}$.

Proof:

$$\int_0^\infty (y-w)dF(w) = y - \int_0^\infty wdF(w) = y - \overline{D}.$$

□

We prove by induction and begin with solving the problem from the last period and proceeding backwards. The following state vectors are valid for the N^{th} and $(N+1)^{st}$ periods:

$$\tilde{x}_N^t = \begin{pmatrix} x_N \\ q_{N-1} \end{pmatrix}$$

$$\tilde{x}_{N+1}^t = \begin{pmatrix} x_N + q_N - W_N \\ q_N \end{pmatrix} = \begin{pmatrix} y_{N-1} - W_{N-1} \\ y_{N-1} - x_{N-1} \end{pmatrix}.$$

Also, cost-to-go function at the last period, $J_N(\tilde{x}_N)$, is in the following form:

$$\begin{aligned} J_N(\tilde{x}_N) &= \min_{q_N \geq 0}\{c_N q_N + L(x_N + q_N) + \gamma\alpha\Delta c_{N+1} E_{W_N}\{\min(q_N, x_N + q_N - W_N)^+\} \\ &\quad + \gamma c_{N+1} E_{W_N}\{\max(W_N - x_N - q_N, 0)\} + \alpha\Delta c_N\{\min(x_N, q_{N-1})\}^+\}. \end{aligned}$$

In terms of y_N, $J_N(\tilde{x}_N)$ can be written as,

$$\begin{aligned} J_N(\tilde{x}_N) &= \min_{y_N \geq x_N}\{c_N y_N + L(y_N) + \gamma\alpha\Delta c_{N+1} E_{W_N}\{\min(y_N - x_N, y_N - W_N)\}^+ \\ &\quad + \gamma c_{N+1} E_{W_N}\{\max(W_N - y_N, 0)\}\} + \alpha\Delta c_N\{\min(x_N, q_{N-1})\}^+ - c_N x_N. \end{aligned}$$

Let

$$\begin{aligned} \tilde{G}_N(y_N|x_N) &= c_N y_N + L(y_N) + \gamma\alpha\Delta c_{N+1} \underbrace{E_{W_N}\{\min(y_N - x_N, y_N - W_N)\}^+}_{A} \\ &\quad + \gamma c_{N+1} \underbrace{E_{W_N}\{\max(W_N - y_N, 0)\}}_{B}. \end{aligned}$$

By using Lemma 3, we can write A as follows:

$$A = \int_0^{y_N} \min(y_N - x_N, y_N - w_N) dF(w_N) = \Phi(y_N) - \Phi(x_N).$$

By using Lemma 2, B can be written as:

$$B = \int_{y_N}^{\infty} (w_N - y_N) dF(w_N) = \overline{D}_N - y_N + \Phi(y_N)$$

After all these calculations the $\tilde{G}_N(y_N|x_N)$ will be the following:

$$\begin{aligned} \tilde{G}_N(y_N|x_N) &= c_N y_N + L(y_N) + \gamma\alpha\Delta c_{N+1}(\Phi(y_N) - \Phi(x_N)) \\ &\quad + \gamma c_{N+1}(\overline{D}_N - y_N + \Phi(y_N)) \\ &= (c_N - \gamma c_{N+1}) y_N + L(y_N) + \gamma\Phi(y_N)(\alpha\Delta c_{N+1} + c_{N+1}) + \gamma c_{N+1}\overline{D}_N \\ &\quad - \gamma\alpha\Delta c_{N+1}\Phi(x_N). \end{aligned}$$

Therefore,

$$\begin{aligned} J_N(\tilde{x}_N) &= \min_{y_N \geq x_N}\{G_N(y_N)\} + \gamma c_{N+1}\overline{D}_N - \gamma\alpha\Delta c_{N+1}\Phi(x_N) \\ &\quad + \alpha\Delta c_N\{\min(x_N, q_{N-1})\}^+ - c_N x_N \end{aligned}$$

where,

$$G_N(y_N) = (c_N - \gamma c_{N+1}) y_N + L(y_N) + \gamma\Phi(y_N)(\alpha\Delta c_{N+1} + c_{N+1}).$$

In order to conclude that $G_N(y_N)$ is convex in y_N, we need to check the second order condition.

$$\frac{\partial G_N(y_N)}{\partial y_N} = (c_N - \gamma c_{N+1}) + (h_N + b_N)F(y_N) - b_N + \gamma F(y_N)(\alpha\Delta c_{N+1} + c_{N+1}),$$

$$\frac{\partial G_N(y_N)^2}{\partial^2 y_N} = f(y_N)(h_N + b_N + \gamma\alpha\Delta c_{N+1} + \gamma c_{N+1}).$$

Therefore, if $(h_N + b_N + \gamma\alpha\Delta c_{N+1} + \gamma c_{N+1}) \geq 0$, $G_N(y_N)$ is convex in y_N. $G_N(y_N)$ attains the minimum value at y_N^* which can be found by,

$$\frac{\partial G_N(y_N)}{\partial y_N} = (c_N - \gamma c_{N+1}) + (h_N + b_N)F(y_N) - b_N + \gamma F(y_N)(\alpha\Delta c_{N+1} + c_{N+1}) = 0.$$

Then,

$$F(y_N^*) = \frac{b_N + \gamma c_{N+1} - c_N}{h_N + b_N + \gamma\alpha\Delta c_{N+1} + \gamma c_{N+1}}.$$

And optimal order-up-to level for the N^{th} period is:

$$y_N^* = F^{-1}\left(\frac{b_N + \gamma c_{N+1} - c_N}{h_N + b_N + \gamma\alpha\Delta c_{N+1} + \gamma c_{N+1}}\right).$$

In order to show that $J_N(\tilde{x}_N)$ is convex in $\tilde{x}_N$, we have to prove that it is convex in x_N and q_{N-1}. For this purpose, we have to calculate the Hessian of $J_N(\tilde{x}_N)$ and check if it is positive-definite or not.

$$H(J_N(\tilde{x}_N) = \begin{bmatrix} \frac{\partial^2 J(\tilde{x}_N)}{\partial {x_N}^2} & \frac{\partial^2 J(\tilde{x}_N)}{\partial x_N \partial q_{N-1}} \\ \frac{\partial^2 J(\tilde{x}_N)}{\partial q_{N-1}\partial x_N} & \frac{\partial^2 J(\tilde{x}_N)}{\partial {q_{N-1}}^2} \end{bmatrix}$$

$$\frac{\partial^2 J_N(\tilde{x}_N)}{\partial {x_N}^2} = \frac{\partial}{\partial x_N}(-\gamma\alpha\Delta c_{N+1}F(x_N)) = -\gamma\alpha\Delta c_{N+1}f(x_N) > 0$$

$$\frac{\partial^2 J_N(\tilde{q}_{N-1})}{\partial {q_{N-1}}^2} = \frac{\partial^2 J_N(\tilde{x}_N)}{\partial x_N \partial q_{N-1}} = \frac{\partial^2 J_N(\tilde{x}_N)}{\partial q_{N-1}\partial x_N} = 0$$

In order to understand $H((J_N(\tilde{x}_N))$ is positive definite we have to find out if $z^t H((J_N(\tilde{x}_N))z > 0$ for all $z \in \Re^2 > 0$ $z = (u_1 \;\; u_2)^t$,

$$\begin{aligned} z^t H((J_N(\tilde{x}_N))z &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} -\gamma\alpha\Delta c_{N+1}f(x_N) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= -\gamma\alpha\Delta c_{N+1}f(x_N)u_1^2 \geq 0 \end{aligned}$$

since $\Delta c_{N+1} \leq 0$. Thus $J_N(\tilde{x}_N)$ is convex in q_{N-1} and x_N.

For the $(N-1)^{st}$ period, the state vectors for the $(N-1)^{st}$ and the N^{th} periods and the cost-to-go function can be written as follows:

$$\tilde{x}^t_{N-1} = \begin{pmatrix} x_{N-1} \\ q_{N-2} \end{pmatrix}$$

$$\tilde{x}^t_N = \begin{pmatrix} x_{N-1} + q_{N-1} - w_{N-1} \\ q_{N-1} \end{pmatrix} = \begin{pmatrix} y_{N-1} - w_{N-1} \\ y_{N-1} - x_{N-1} \end{pmatrix},$$

$$\begin{aligned} J_{N-1}(\tilde{x}_{N-1}) &= \min_{y_{N-1} \geq x_{N-1}} \{c_{N-1} y_{N-1} + L(y_{N-1}) + \gamma E_{W_{N-1}}\{J_N(\tilde{x}_N)\}\} \\ &\quad + \alpha \Delta c_{N-1} \{\min(x_{N-1}, q_{N-2})\}^+ - c_{N-1} x_{N-1}. \end{aligned}$$

The cost-to-go function for the N^{th} period in terms of $G_N(y_N)$ is:

$$J_N(\tilde{x}_N) = \begin{cases} G_N(y_{N-1} - W_{N-1}) & \\ -\gamma\alpha\Delta c_{N+1}\Phi(y_{N-1} - W_{N-1}) & \\ +\alpha\Delta c_N \min(y_{N-1} - W_{N-1}, y_{N-1} - x_{N-1}) & \\ -c_N(y_{N-1} - W_{N-1}) + \gamma c_{N+1}\overline{D}_N & \text{if } y_{N-1} - W_{N-1} \geq 0 \\ & \text{and } y^*_N \leq y_{N-1} - W_{N-1} \\ G_N(y^*_N) - \gamma\alpha\Delta c_{N+1}\Phi(y_{N-1} - W_{N-1}) & \\ +\alpha\Delta c_N \min(y_{N-1} - W_{N-1}, y_{N-1} - x_{N-1}) & \\ -c_N(y_{N-1} - W_{N-1}) + \gamma c_{N+1}\overline{D}_N & \text{if } y^*_N \geq y_{N-1} - W_{N-1} \geq 0 \\ & \\ G_N(y^*_N) - \gamma\alpha\Delta c_{N+1}\Phi(y_{N-1} - W_{N-1}) & \\ -c_N(y_{N-1} - W_{N-1}) + \gamma c_{N+1}\overline{D}_N & \text{if } y_{N-1} - W_{N-1} < 0 \end{cases}$$

And we have

$$\begin{aligned} E_{W_{N-1}}\{J_N(\tilde{x}_N)\} &= \int_0^\infty J_N(\tilde{x}_N) dF(w_{N-1}) \\ &= \int_0^{y_{N-1} - y^*_N} G_N(y_{N-1} - w_{N-1}) dF(w_{N-1}) \\ &\quad + \int_{y_{N-1} - y^*_N}^\infty G_N(y^*_N) dF(w_{N-1}) \end{aligned}$$

$$
\begin{aligned}
&- \gamma\alpha\Delta c_{N+1} \int_0^{\infty} \Phi(y_{N-1} - w_{N-1}) dF(w_{N-1}) \\
&+ \alpha\Delta c_N \int_0^{y_{N-1}} \min(y_{N-1} - w_{N-1}, y_{N-1} - x_{N-1}) dF(w_{N-1}) \\
&- c_N \int_0^{\infty} (y_{N-1} - w_{N-1}) dF(w_{N-1}) + \gamma c_{N+1} \overline{D}_N.
\end{aligned}
$$

By the use of Lemmas 3 and 4:

$$
\begin{aligned}
E_{W_{N-1}}\{J_N(\tilde{x}_N)\} &= \int_0^{y_{N-1}-y_N^*} G_N(y_{N-1} - w_{N-1}) dF(w_{N-1}) \\
&\quad + \int_{y_{N-1}-y_N^*}^{\infty} G_N(y_N^*) dF(w_{N-1}) \\
&\quad - \gamma\alpha\Delta c_{N+1} \int_0^{\infty} \Phi(y_{N-1} - w_{N-1}) dF(w_{N-1}) \\
&\quad + \alpha\Delta c_N \left(\Phi(y_{N-1}) - \Phi(x_{N-1})\right) \\
&\quad - c_N(y_{N-1} - \overline{D}_{N-1}) + \gamma c_{N+1} \overline{D}_N \\
&= \int_0^{y_{N-1}-y_N^*} G_N(y_{N-1} - w_{N-1}) dF(w_{N-1}) \\
&\quad + G_N(y_N^*) - G_N(y_N^*) F(y_{N-1} - y_N^*) \\
&\quad - \gamma\alpha\Delta c_{N+1} \int_0^{\infty} \Phi(y_{N-1} - w_{N-1}) dF(w_{N-1}) \\
&\quad + \alpha\Delta c_N \Phi(y_{N-1}) - \alpha\Delta c_N \Phi(x_{N-1}) \\
&\quad - c_N y_{N-1} + c_N \overline{D}_{N-1} + \gamma c_{N+1} \overline{D}_N.
\end{aligned}
$$

The cost-to-go function for the $(N-1)^{st}$ period is the following:

$$
\begin{aligned}
J_{N-1}(\tilde{x}_{N-1}) &= \min_{y_{N-1} \geq x_{N-1}} \{(c_{N-1} - \gamma c_N) y_{N-1} + L(y_{N-1}) \\
&\quad + \gamma \int_0^{y_{N-1}-y_N^*} G_N(y_{N-1} - w_{N-1}) dF(w_{N-1}) \\
&\quad - \gamma G_N(y_N^*) F(y_{N-1} - y_N^*) \\
&\quad - \gamma^2 \alpha\Delta c_{N+1} \int_0^{+\infty} \Phi(y_{N-1} - w_{N-1}) dF(w_{N-1}) \\
&\quad + \gamma\alpha\Delta c_N \Phi(y_{N-1})\} - \gamma\alpha\Delta c_N \Phi(x_{N-1}) \\
&\quad + \gamma c_N \overline{D}_{N-1} + \gamma^2 c_{N+1} \overline{D}_N + \gamma G_N(y_N^*) \\
&\quad + \alpha\Delta c_{N-1} \{\min(x_{N-1}, q_{N-2})\}^+ - c_{N-1} x_{N-1}.
\end{aligned}
$$

Hence

$$
J_{N-1}(\tilde{x}_{N-1}) = \min_{y_{N-1} \geq x_{N-1}} \{G_{N-1}(y_{N-1})\}
$$

$$
\begin{aligned}
&- \gamma\alpha\Delta c_N \Phi(x_{N-1}) + \gamma c_N \overline{D}_{N-1} + \gamma^2 c_{N+1}\overline{D}_N + \gamma G_N(y_N^*) \\
&+ \alpha\Delta c_{N-1}\{\min(x_{N-1}, q_{N-2})\}^+ - c_{N-1}x_{N-1},
\end{aligned}
$$

where

$$
\begin{aligned}
G_{N-1}(y_{N-1}) \quad = \quad & (c_{N-1} - \gamma c_N)y_{N-1} + L(y_{N-1}) \\
& + \gamma \int_0^{y_{N-1}-y_N^*} G_N(y_{N-1} - w_{N-1})dF(w_{N-1}) \\
& - \gamma G_N(y_N^*)F(y_{N-1} - y_N^*) \\
& - \gamma^2\alpha\Delta c_{N+1} \int_0^\infty \Phi(y_{N-1} - w_{N-1})dF(w_{N-1}) \\
& + \gamma\alpha\Delta c_N \Phi(y_{N-1}).
\end{aligned}
$$

Next we need to show that $G_{N-1}(y_{N-1})$ is convex in y_{N-1} so need to check the second order condition. Recall the Leibnitz Rule.

Leibnitz Rule

$$
\frac{\partial}{\partial z}\int_{\alpha(z)}^{\beta(z)} f(x,z)dx = \int_{\alpha(z)}^{\beta(z)} \frac{\partial f}{\partial z}dx + f(\beta(z),z)\frac{\partial \beta(z)}{\partial z} - f(\alpha(z),z)\frac{\partial \alpha(z)}{\partial z}.
$$

$$
\begin{aligned}
\frac{\partial G_{N-1}(y_{N-1})}{\partial y_{N-1}} \quad = \quad & (\gamma c_N - c_{N-1}) + (h_{N-1} + b_{N-1})F(y_{N-1}) - b_{N-1} \\
& + \gamma G_N(y_N^*)f(y_{N-1} - y_N^*) \\
& + \gamma \int_0^{y_{N-1}-y_N^*} G'_N(y_{N-1} - w_{N-1})f(w_{N-1})dw_{N-1} \\
& - \gamma G_N(y_N^*)f(y_{N-1} - y_N^*) \\
& - \gamma^2\alpha\Delta c_{N+1} \int_0^\infty F(y_{N-1} - w_{N-1})f(w_{N-1})dw_{N-1} \\
& + \gamma\alpha\Delta c_N F(y_{N-1}) \\
= \quad & (c_{N-1} - \gamma c_N) + (h_{N-1} + b_{N-1} + \gamma\alpha\Delta c_N)F(y_{N-1}) - b_{N-1} \\
& + \gamma \int_0^{y_{N-1}-y_N^*} G'_N(y_{N-1} - w_{N-1})f(w_{N-1})dw_{N-1} \\
& - \gamma^2\alpha\Delta c_{N+1} \int_0^\infty F(y_{N-1} - w_{N-1})f(w_{N-1})dw_{N-1}
\end{aligned}
$$

$$
\frac{\partial^2 G_{N-1}(y_{N-1})}{\partial y_{N-1}^2} \quad = \quad (h_{N-1} + b_{N-1} + \gamma\alpha\Delta c_N)f(y_{N-1})
$$

$$+ \gamma \int_0^{y_{N-1}-y_N^*} G_N^{''}(y_{N-1} - w_{N-1}) f(w_{N-1}) dw_{N-1}$$
$$+ \gamma G_N^{'}(y_N^*) f(y_{N-1} - y_N^*)$$
$$- \gamma^2 \alpha \Delta c_{N+1} \int_0^{\infty} f(y_{N-1} - w_{N-1}) f(w_{N-1}) dw_{N-1}$$

where, $G_N^{'}$ and $G_N^{''}$ are the first and second derivatives of G_N with respect to y_N respectively. Note that $G_N^{'}(y_N^*) = 0$. Since $G_N(y_N)$ is convex wrt y_N, $G_N^{''}(y_N) \geq 0$. Therefore $G_N^{''}(y_{N-1} - w_{N-1})$ is nonnegative. Also since f is a pdf, $\int_0^{\infty} f(y_{N-1} - w_{N-1}) f(w_{N-1}) dw_{N-1} \geq 0$. These all make

$$\frac{\partial^2 G_{N-1}(y_{N-1})}{\partial y_{N-1}^2} \geq 0$$

and thus convex in y_{N-1} if $(h_{N-1} + b_{N-1} + \gamma \alpha \Delta c_N) \geq 0$.

In order to understand $H(J_{N-1}(\tilde{x}_{N-1}))$ is positive definite we have to find out if $z^* H(J_{N-1}(\tilde{x}_{N-1}))z > 0$ for all $z \in \Re^2 > 0$ $z = (u_1 \ \ u_2)^t$

$$\begin{aligned} z^t H((J_{N-1}(\tilde{x}_{N-1}))z &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} -\gamma \alpha \Delta c_N f(x_{N-1}) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= -\gamma \alpha \Delta c_N f(x_{N-1}) u_1^2 \geq 0 \end{aligned}$$

Therefore, $J_{N-1}(\tilde{x}_{N-1})$ is convex in x_{N-1} and q_{N-2} since $\Delta c_N \leq 0$. Up to now, we show that *(i)* and *(ii)* hold for the N^{th} and $(N-1)^{st}$ periods. Also we show that *(iv)* holds. Part *(iii)* follows from the convexity of the $G(.)$ functions. From the induction hypothesis, assume that *(i)* and *(ii)* holds for periods $k+1,\ k+2, ..., N$. For the k^{th} and the $(k+1)^{st}$ periods we have the following state vectors respectively:

$$\tilde{x}_k^t = \begin{pmatrix} x_k \\ q_{k-1} \end{pmatrix}$$

$$\tilde{x}_{k+1}^t = \begin{pmatrix} x_k + q_k - W_k \\ q_k \end{pmatrix} = \begin{pmatrix} y_k - W_k \\ y_k - x_k \end{pmatrix}.$$

Cost-to-go function at the k^{th} period is:

$$\begin{aligned} J_k(\tilde{x}_k) = & \min_{y_k \geq x_k} \{c_k y_k + L(y_k) + \gamma E_{W_k}\{J_{k+1}(\tilde{x}_{k+1})\}\} + \alpha \Delta c_k \{\min(x_k, q_{k-1})\}^+ \\ & - c_k x_k. \end{aligned}$$

By inductive assumption,

$$\begin{aligned} J_{k+1}(\tilde{x}_{k+1}) &= \min_{y_{k+1}\geq x_{k+1}} \{G_{k+1}(y_{k+1})\} - \gamma\alpha\Delta c_{k+2}\Phi(x_{k+1}) \\ &+ \alpha\Delta c_{k+1}\{\min(x_{k+1}, q_k)\}^{+} - c_{k+1}x_{k+1} \\ &+ \left(\sum_{i=k+1}^{i=N-1} \gamma^{i+1-k}\left(c_{i+1}\overline{D}_i + G_{i+1}(y^*_{i+1})\right)\right) + \gamma^{N+1-k-1}c_{N+1}\overline{D}_N. \end{aligned}$$

Expected cost-to-go function in the k^{th} period,

$$\begin{aligned} E_{W_k}\{J_{k+1}(\tilde{x}_{k+1})\} &= \int_0^{\infty} J_{k+1}(\tilde{x}_{k+1})dF(w_k) \\ &= \int_0^{y_k-y^*_{k+1}} G_{k+1}(y_k-w_k)dF(w_k) \\ &+ \int_{y_k-y^*_{k+1}}^{\infty} G_{k+1}(y^*_{k+1})dF(w_k) \\ &- \gamma\alpha\Delta c_{k+2}\int_0^{\infty}\Phi(y_k-w_k)dF(w_k) \\ &+ \alpha\Delta c_{k+1}\int_0^{y_k}\min(y_k-w_k, y_k-x_k)dF(w_k) \\ &- c_{k+1}\int_0^{\infty}(y_k-w_k)dF(w_k) \\ &+ \left(\sum_{i=k+1}^{i=N-1}\gamma^{i+1-k}\left(c_{i+1}\overline{D}_i + G_{i+1}(y^*_{i+1})\right)\right) + \gamma^{N-k}c_{N+1}\overline{D}_N. \end{aligned}$$

From Lemmas 3 and 4 we can simplify the above equation in the following way:

$$\begin{aligned} E_{W_k}\{J_{k+1}(\tilde{x}_{k+1})\} &= \int_0^{y_k-y^*_{k+1}} G_{k+1}(y_k-w_k)dF(w_k) \\ &+ G_{k+1}(y^*_{k+1}) - G_{k+1}(y^*_{k+1})F(y_k-y^*_{k+1}) \\ &- \gamma\alpha\Delta c_{k+2}\int_0^{\infty}\Phi(y_k-w_k)dF(w_k) \\ &+ \alpha\Delta c_{k+1}(\Phi(y_k)-\Phi(x_k)) \\ &- c_{k+1}(y_k-\overline{D}_k) + \left(\sum_{i=k+1}^{i=N-1}\gamma^{i-k}\left(c_{i+1}\overline{D}_i + G_{i+1}(y^*_{i+1})\right)\right) \\ &+ \gamma^{N-k}c_{N+1}\overline{D}_N \\ &= \int_0^{y_k-y^*_{k+1}} G_{k+1}(y_k-w_k)dF(w_k) - G_{k+1}(y^*_{k+1})F(y_k-y^*_{k+1}) \\ &- \gamma\alpha\Delta c_{k+2}\int_0^{\infty}\Phi(y_k-w_k)dF(w_k) + \alpha\Delta c_{k+1}\Phi(y_k) - c_{k+1}y_k \end{aligned}$$

$$- \alpha\Delta c_{k+1}\Phi(x_k) + c_{k+1}\overline{D}_k + G_{k+1}(y^*_{k+1})$$
$$+ \left(\sum_{i=k+1}^{i=N-1} \gamma^{i-k} \left(c_{i+1}\overline{D}_i + G_{i+1}(y^*_{i+1}) \right) \right) + \gamma^{N-k} c_{N+1}\overline{D}_N.$$

Cost-to-go function at the k^{th} period in terms of G_k is the following:

$$\begin{aligned} J_k(\tilde{x}_k) &= \min_{y_k \geq x_k} \{G_k(y_k)\} - \gamma\alpha\Delta c_{k+1}\Phi(x_k) + \alpha\Delta c_k\{\min(x_k, q_{k-1})\}^+ - c_k x_k \\ &\quad + \left(\sum_{i=k}^{i=N-1} \gamma^{i-k+1} \left(c_{i+1}\overline{D}_i + G_{i+1}(y^*_{i+1}) \right) \right) + \gamma^{N-k+1} c_{N+1}\overline{D}_N \end{aligned}$$

where

$$\begin{aligned} G_k(y_k) &= (c_k - \gamma c_{k+1})y_k + L(y_k) + \gamma \int_0^{y_k - y^*_{k+1}} G_{k+1}(y_k - w_k) dF(w_{N-2}) \\ &\quad - \gamma G_{k+1}(y^*_{k+1}) F(y_k - y^*_{k+1}) - \gamma^2 \alpha\Delta c_{k+2} \int_0^{\infty} \Phi(y_k - w_k) dF(w_k) \\ &\quad + \gamma\alpha\Delta c_{k+1}\Phi(y_k). \end{aligned}$$

The convexity of the $G_k(y_k)$ in y_k is shown similar to the $(N-1)^{st}$ period. Finally we have,

$$\begin{aligned} z^* H((J_k(\tilde{x}_k))z &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} -\gamma\alpha\Delta c_{k+1} f(x_k) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= -\gamma\alpha\Delta c_{k+1} f(x_k) u_1^2 \geq 0, \end{aligned}$$

since $\Delta c_{k+1} \leq 0$. Thus $J_k(\tilde{x}_k)$ is convex in x_k and q_{k-1} and order-up-to policy is optimal for the retailer.

□

Proof of Theorem 2: Similarly, we prove by induction. The state vectors for the N^{th} and $(N+1)^{st}$ periods and the cost-to-go function of the N^{th} period are the following:

$$\tilde{x}^t_N = \begin{pmatrix} x_N \\ q_{N-2} \\ q_{N-1} \end{pmatrix}$$

$$\tilde{x}^t_{N+1} = \begin{pmatrix} x_N + q_N - w_N \\ q_{N-1} \\ q_N \end{pmatrix} = \begin{pmatrix} y_N - w_N \\ q_{N-1} \\ y_N - x_N \end{pmatrix},$$

$$\begin{aligned}
J_N(\tilde{x}_N) &= \min_{q_N \geq 0}\{c_N q_N + L_N(x_N + q_N) + \alpha \Delta c_N \{\min(x_N, q_{N-2} + q_{N-1})^+\} \\
&\quad + \gamma \alpha \Delta c_{N+1} E_{W_N}\{\min(x_N + q_N - W_N, q_{N-1} + q_N)^+\} \\
&\quad + \gamma c_{N+1} E_{W_N}\{\max(W_N - x_N - q_N, 0)\}\} \\
&= \min_{y_N \geq x_N}\{c_N y_N + L_N(y_N) \\
&\quad + \gamma \alpha \Delta c_{N+1} E_{W_N}\{\min(y_N - W_N, q_{N-1} + y_N - x_N)^+\} \\
&\quad + \gamma c_{N+1} E_{W_N}\{\max(W_N - y_N, 0)\}\} + \alpha \Delta c_N \{\min(x_N, q_{N-1} + q_{N-2})\}^+ \\
&\quad - c_N x_N.
\end{aligned}$$

Let $I = E_{W_N}\{\min(y_N - W_N, q_{N-1} + y_N - x_N)^+\}$ and $II = E_{W_N}\{\max(W_N - y_N, 0)\}\}$. Then,

$$\begin{aligned}
I &= \int_0^{x_N - q_{N-1}} (y_N + q_{N-1} - x_N) dF(w_N) + \int_{x_N - q_{N-1}}^{y_N} (y_N - w_N) dF(w_N) \\
&= (y_N + q_{N-1} - x_N) F(x_N - q_{N-1}) + y_N \left(F(y_N) - F(x_N - q_{N-1})\right) \\
&\quad - \int_{x_N - q_{N-1}}^{y_N} w_N dF(w_N) \\
&= y_N F(x_N - q_{N-1}) + (q_{N-1} - x_N) F(x_N - q_{N-1}) \\
&\quad + y_N F(y_N) - y_N F(x_N - q_{N-1}) \\
&\quad - \left(\int_0^{y_N} w_N dF(w_N) - \int_0^{x_N - q_{N-1}} w_N dF(w_N)\right).
\end{aligned}$$

By using Lemma 1, I will be obtained as: $I = \Phi(y_N) - \Phi(x_N - q_{N-1})$
By using Lemma 4, II will be obtained as: $II = \overline{D}_N - y_N + \Phi(y_N)$
Therefore, cost-to-go function in the N^{th} period is:

$$\begin{aligned}
J_N(\tilde{x}_N) &= \min_{y_N \geq x_N}\{c_N y_N + L(y_N) + \alpha \Delta c_N \{\min(x_N, q_{N-1} + q_{N-2})^+\} \\
&\quad + \gamma \alpha \Delta c_{N+1}(\Phi(y_N) - \Phi(x_N - q_{N-1})) \\
&\quad + \gamma c_{N+1}(\overline{D}_N - y_N + \Phi(y_N))\} - c_N x_N \\
&= \min_{y_N \geq x_N}\{(c_N - \gamma c_{N+1}) y_N + L(y_N) + \gamma \Phi(y_N)(\alpha \Delta c_{N+1} + c_{N+1})\} \\
&\quad - \gamma \alpha \Delta c_{N+1} \Phi(x_N - q_{N-1}) + \gamma c_{N+1} \overline{D}_N \\
&\quad + \alpha \Delta c_N \{\min(x_N, q_{N-1} + q_{N-2})^+ - c_N x_N.
\end{aligned}$$

Cost-to-go function in terms of $G_N(y_N)$ is the following:

$$\begin{aligned}
J_N(\tilde{x}_N) &= \min_{y_N \geq x_N}\{G_N(y_N)\} - \gamma \alpha \Delta c_{N+1} \Phi(x_N - q_{N-1}) + \gamma c_{N+1} \overline{D}_N \\
&\quad + \alpha \Delta c_N \min(x_N, q_{N-1} + q_{N-2})^+ - c_N x_N
\end{aligned}$$

where,

$$G_N(y_N) = (c_N - \gamma c_{N+1})y_N + L(y_N) + \gamma\Phi(y_N)(\alpha\Delta c_{N+1} + c_{N+1}).$$

We check the second order condition for convexity.

$$\frac{\partial G_N(y_N)}{\partial y_N} = (c_N - \gamma c_{N+1}) + (h_N + b_N)F(y_N) - b_N + \gamma F(y_N)(\alpha\Delta c_{N+1} + c_{N+1}),$$

$$\frac{\partial G_N(y_N)^2}{\partial^2 y_N} = f(y_N)(h_N + b_N + \gamma\alpha\Delta c_{N+1} + c_{N+1}).$$

If $(h_N + b_N + \gamma\alpha\Delta c_{N+1} + \gamma c_{N+1}) \geq 0$, $f(y_N)(h_N + b_N + \gamma\alpha\Delta c_{N+1} + \gamma c_{N+1}) \geq 0$. Therefore $G_N(y_N)$ is convex in y_N. And, order-up-to level for the N^{th} can be found by,

$$\frac{\partial G_N(y_N)}{\partial y_N} = (c_N - \gamma c_{N+1}) + (h+b)F(y_N) - b + F(y_N)(\alpha\Delta c_{N+1} + c_{N+1}) = 0,$$

$$F(y_N^*) = \frac{b_N - c_N + \gamma c_{N+1}}{h_N + b_N + \gamma\alpha\Delta c_{N+1} + \gamma c_{N+1}},$$
$$y_N^* = F^{-1}\left(\frac{b_N - c_N + \gamma c_{N+1}}{h_N + b_N + \gamma\alpha\Delta c_{N+1} + \gamma c_{N+1}}\right).$$

In order to show the convexity of $J_N(\tilde{x}_N)$, we need to show that the Hessian Matrix of J_N is positive definite. The Hessian Matrix:

$$H(J_N(\tilde{x}_N)) = \begin{bmatrix} \frac{\partial^2 J_N(\tilde{x}_N)}{\partial {x_N}^2} & \frac{\partial^2 J_N(\tilde{x}_N)}{\partial x_N \partial q_{N-1}} & \frac{\partial^2 J_N(\tilde{x}_N)}{\partial x_N \partial q_{N-2}} \\ \frac{\partial^2 J_N(\tilde{x}_N)}{\partial q_{N-1} \partial x_N} & \frac{\partial^2 J_N(\tilde{x}_N)}{\partial {q_{N-1}}^2} & \frac{\partial^2 J_N(\tilde{x}_N)}{\partial q_{N-1} \partial q_{N-2}} \\ \frac{\partial^2 J_N(\tilde{x}_N)}{\partial q_{N-2} \partial x_N} & \frac{\partial^2 J_N(\tilde{x}_N)}{\partial q_{N-2} \partial q_{N-1}} & \frac{\partial^2 J_N(\tilde{x}_N)}{\partial {q_{N-2}}^2} \end{bmatrix}.$$

$$\frac{\partial^2 J_N(\tilde{x}_N))}{\partial {x_N}^2} = \frac{\partial^2 J_N(\tilde{x}_N)}{\partial {q_{N-1}}^2} = -\gamma\alpha\Delta c_{N+1} f(x_N - q_{N-1}),$$

$$\frac{\partial^2 J_N(\tilde{x}_N))}{\partial x_N \partial q_{N-1}} = \frac{\partial^2 J_N(\tilde{x}_N))}{\partial q_{N-1} \partial x_N} = \gamma\alpha\Delta c_{N+1} f(x_N - q_{N-1}),$$

$$\frac{\partial^2 J_N(\tilde{x}_N))}{\partial x_N \partial q_{N-2}} = \frac{\partial^2 J_N(\tilde{x}_N))}{\partial q_{N-1} \partial q_{N-2}} = \frac{\partial^2 J_N(\tilde{x}_N))}{\partial q_{N-2} \partial x_N} = \frac{\partial^2 J_N(\tilde{x}_N))}{\partial q_{N-2} \partial q_{N-1}} = \frac{\partial^2 J_N(\tilde{x}_N))}{\partial {q_{N-2}}^2} = 0.$$

$$H_N = H(J_N(\tilde{x}_N))) = \begin{bmatrix} -\gamma\alpha\Delta c_{N+1} f(x_N - q_{N-1}) & \gamma\alpha\Delta c_{N+1} f(x_N - q_{N-1}) & 0 \\ \gamma\alpha\Delta c_{N+1} f(x_N - q_{N-1}) & -\gamma\alpha\Delta c_{N+1} f(x_N - q_{N-1}) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Assume $u_1 \geq 0, u_2 \geq 0, u_3 \geq 0$ and $z = (u_1 \;\; u_2 \;\; u_3)^t$. The Hessian is:

$$z^t H_N z \;\; = -\gamma\alpha\Delta c_{N+1} f(x_N - q_{N-1})(u_1 - u_2)^2 \geq 0,$$

since $\Delta c_{N+1} \geq 0$. Therefore $J_N(\tilde{x}_N)$ is convex in x_N, q_{N-2} and q_{N-1}.

If we continue to solve the problem for the $(N-1)^{st}$ period we obtain the following state vectors for the $(N-1)^{st}$ and N^{th} and cost-to-go function for the $(N-1)^{st}$ period:

$$\tilde{x}^t_{N-1} = \begin{pmatrix} x_{N-1} \\ q_{N-3} \\ q_{N-2} \end{pmatrix}$$

$$\tilde{x}^t_N = \begin{pmatrix} x_{N-1} + q_{N-1} - w_{N-1} \\ q_{N-2} \\ q_{N-1} \end{pmatrix} = \begin{pmatrix} y_{N-1} - w_{N-1} \\ q_{N-2} \\ y_{N-1} - x_{N-1} \end{pmatrix}$$

$$\begin{aligned} J_{N-1}(\tilde{x}_{N-1}) &= \min_{y_{N-1} \geq x_{N-1}} \{c_{N-1} y_{N-1} + L(y_{N-1}) + \gamma E_{W_{N-1}}\{J_N(\tilde{x}_N)\}\} \\ &\quad + \alpha\Delta c_{N-1} \min(x_{N-1}, q_{N-3} + q_{N-2})^+ - c_{N-1} x_{N-1}. \end{aligned}$$

The expected cost-to-go function in the $(N-1)^{st}$ is the following:

$$\begin{aligned} E_{W_{N-1}}\{J_N(\tilde{x}_N)\} &= \int_0^\infty J_N(\tilde{x}_N) dF(w_{N-1}) \\ &= \int_0^{y_{N-1} - y^*_N} G_N(y_{N-1} - w_{N-1}) dF(w_{N-1}) \\ &\quad + \int_{y_{N-1} - y^*_N}^\infty G_N(y^*_N) dF(w_{N-1}) \\ &\quad - \gamma\alpha\Delta c_{N+1} \int_0^\infty \Phi(x_{N-1} - w_{N-1}) dF(w_{N-1}) \\ &\quad + \alpha\Delta c_N \int_0^{y_{N-1}} \min(y_{N-1} - w_{N-1}, y_{N-1} - x_{N-1} + q_{N-2}) dF(w_{N-1}) \\ &\quad - c_N \int_0^\infty (y_{N-1} w_{N-1}) dF(w_{N-1}) + \gamma c_{N+1} \overline{D}_N. \end{aligned}$$

By using Lemmas 3 and 4 and after a few arrangements:

$$\begin{aligned} E_{W_{N-1}}\{J_N(\tilde{x}_N)\} &= \int_0^{y_{N-1} - y^*_N} G_N(y_{N-1} - w_{N-1}) dF(w_{N-1}) \\ &\quad + G_N(y^*_N)(1 - F(y_{N-1} - y^*_N)) \end{aligned}$$

$$
\begin{aligned}
& - \gamma\alpha\Delta c_{N+1} \int_0^\infty \Phi(x_{N-1} - w_{N-1}) dF(w_{N-1}) \\
& + \alpha\Delta c_N(\Phi(y_{N-1}) - \Phi(x_{N-1} - q_{N-2})) - c_N y_{N-1} + c_N \overline{D}_{N-1} \\
& + \gamma c_{N+1} \overline{D}_N \\
= & \int_0^{y_{N-1}-y_N^*} G_N(y_{N-1} - w_{N-1}) dF(w_{N-1}) \\
& - G_N(y_N^*) F(y_{N-1} - y_N^*) \\
& + \alpha\Delta c_N \Phi(y_{N-1}) - c_N y_{N-1} \\
& - \gamma\alpha\Delta c_{N+1} \int_0^\infty \Phi(x_{N-1} - w_{N-1}) dF(w_{N-1}) \\
& + G_N(y_N^*) - \alpha\Delta c_N \Phi(x_{N-1} - q_{N-2}) + \gamma c_{N+1} \overline{D}_N + c_N \overline{D}_{N-1}.
\end{aligned}
$$

After a few adjustments we have the following cost-to-go function for the $(N-1)^{st}$ period:

$$
\begin{aligned}
J_{N-1}(\tilde{x}_{N-1}) = & \min_{y_{N-1} \geq x_{N-1}} \{G_{N-1}(y_{N-1})\} \\
& - \gamma^2 \alpha\Delta c_{N+1} \int_0^\infty \Phi(x_{N-1} - w_{N-1}) dF(w_{N-1}) \\
& - \gamma\alpha\Delta c_N \Phi(x_{N-1} - q_{N-2}) + \alpha\Delta c_{N-1} \min(x_{N-1}, q_{N-3} + q_{N-2})^+ \\
& + \gamma G_N(y_N^*) + \gamma^2 c_{N+1} \overline{D}_N + \gamma c_N \overline{D}_{N-1} - c_{N-1} x_{N-1}
\end{aligned}
$$

where,

$$
\begin{aligned}
G_{N-1}(y_{N-1}) = & (c_{N-1} - \gamma c_N) y_{N-1} + L(y_{N-1}) \\
& + \gamma \int_0^{y_{N-1}-y_N^*} G_N(y_{N-1} - w_{N-1}) dF(w_{N-1}) \\
& - \gamma G_N(y_N^*) F(y_{N-1} - y_N^*) + \gamma\alpha\Delta c_N \Phi(y_{N-1}).
\end{aligned}
$$

Check the second order condition for convexity:

$$
\begin{aligned}
\frac{\partial G_{N-1}(y_{N-1})}{\partial y_{N-1}} = & (c_{N-1} - \gamma c_N) + (h_{N-1} + b_{N-1}) F(y_{N-1}) - b_{N-1} \\
& + \gamma \int_0^{y_{N-1}-y_N^*} G_N'(y_{N-1} - w_{N-1}) f(w_{N-1}) dw_{N-1} \\
& + \gamma G_N(y_N^*) f(y_{N-1} - y_N^*) - \gamma G_N(y_N^*) f(y_{N-1} - y_N^*) \\
& + \gamma\alpha\Delta c_N F(y_{N-1}) \\
= & (c_{N-1} - \gamma c_N) + (h_{N-1} + b_{N-1} + \gamma\alpha\Delta c_N) F(y_{N-1}) - b_{N-1} \\
& + \gamma \int_0^{y_{N-1}-y_N^*} G_N'(y_{N-1} - w_{N-1}) f(w_{N-1}) dw_{N-1}
\end{aligned}
$$

$$\begin{aligned}\frac{\partial^2 G_{N-1}(y_{N-1})}{\partial y_{N-1}^2} &= (h_{N-1}+b_{N-1}+\gamma\alpha\Delta c_N)f(y_{N-1}) \\ &\quad +\gamma\int_0^{y_{N-1}-y_N^*} G_N^{''}(y_{N-1}-w_{N-1})f(w_{N-1})dw_{N-1} \\ &\quad +\gamma G_N^{'}(y_N^*)f(y_{N-1}-y_N^*) \\ &= (h_{N-1}+b_{N-1}+\gamma\alpha\Delta c_N)f(y_{N-1}) \\ &\quad +\gamma\int_0^{y_{N-1}-y_N^*} G_N^{''}(y_{N-1}-w_{N-1})f(w_{N-1})dw_{N-1},\end{aligned}$$

where, $G_N^{'}$ and $G_N^{''}$ are the first and the second derivatives of $G_N(y_N)$ function wrt y_N respectively.

Note that since G_N is convex $G_N^{''}(y_{N-1}-w_{N-1}) \geq 0$ and $G_N^{'}(y_N^*) = 0$ due to optimality. Additionally, as long as $h_{N-1}+b_{N-1}+\gamma\alpha\Delta c_N \geq 0$ and $h_N + b_N + \gamma\alpha\Delta c_{N+1} + \gamma c_{N+1} \geq 0$ $G_{N-1}(y_{N-1})$ is convex. To prove the convexity of $J_{N-1}(\tilde{x}_{N-1})$, we check the Hessian of $J_{N-1}(\tilde{x}_{N-1})$.

$$\frac{\partial^2 J_{N-1}(\tilde{x}_{N-1})}{\partial {x_{N-1}}^2} = -\gamma^2\alpha\Delta c_{N+1}\int_0^\infty f(x_{N-1}-w_{N-1})dF(w_{N-1}) - \gamma\alpha\Delta c_N f(x_{N-1}-q_{N-2}),$$

$$\frac{\partial^2 J_{N-1}(\tilde{x}_{N-1})}{\partial x_{N-1}\partial q_{N-2}} = \gamma\alpha\Delta c_N f(x_{N-1}-q_{N-2}),$$

$$\frac{\partial^2 J_{N-1}(\tilde{x}_{N-1})}{\partial x_{N-1}\partial q_{N-3}} = \frac{\partial^2 J_{N-1}(\tilde{x}_{N-1})}{\partial q_{N-3}\partial x_{N-1}} = \frac{\partial^2 J_{N-1}(\tilde{x}_{N-1})}{\partial {q_{N-3}}^2} = \frac{\partial^2 J_{N-1}(\tilde{x}_{N-1})}{\partial q_{N-3}\partial q_{N-2}} = \frac{\partial^2 J_{N-1}(\tilde{x}_{N-1})}{\partial q_{N-2}\partial q_{N-3}} = 0,$$

$$\frac{\partial^2 J_{N-1}(\tilde{x}_{N-1})}{\partial q_{N-2}\partial x_{N-1}} = \gamma\alpha\Delta c_N f(x_{N-1}-q_{N-2}),$$

$$\frac{\partial^2 J_{N-1}(\tilde{x}_{N-1})}{\partial {q_{N-2}}^2} = -\gamma\alpha\Delta c_N f(x_{N-1}-q_{N-2}).$$

Assume $u_1 \geq 0$, $u_2 \geq 0$, $u_3 \geq 0$ and $z = (u_1\ u_2\ u_3)^t$. Let $K = \gamma^2\alpha\Delta c_{N+1}\int_0^\infty f(x_{N-1}-w_{N-1})dF(w_{N-1})$ and $A = \gamma\alpha\Delta c_N f(x_{N-1}-q_{N-2})$. Note that $K \leq 0$ and $A \leq 0$ since $\Delta c_{N+1} \leq 0$ and $\Delta c_N \leq 0$ and $f(.) \geq 0$.

$$z^t H_{N-1} z = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} -K-A & 0 & A \\ 0 & 0 & 0 \\ A & 0 & -A \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$
\begin{aligned}
&= \left[\begin{array}{ccc} u_1(-K-A)+u_3A & 0 & (u_1-u_3)A \end{array}\right]\left[\begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array}\right] \\
&= (u_1(-K-A)+u_3A)u_1 + (u_1A-u_3A)u_3 \\
&= -Ku_1^2 - A(u_1-u_3)^2 \geq 0
\end{aligned}
$$

Therefore Hessian Matrix of J_{N-1} is positive definite. Thus $J_{N-1}(\tilde{x}_{N-1})$ is convex in x_{N-1}, q_{N-3} and q_{N-2}.

If we extend the argument for the $(N-2)^{nd}$ period, we have the following state vectors for the $(N-2)^{nd}$ and $(N-1)^{st}$ periods:

$$
\tilde{x}_{N-2}^t = \left(\begin{array}{c} x_{N-2} \\ q_{N-4} \\ q_{N-3} \end{array}\right)
$$

$$
\tilde{x}_{N-1}^t = \left(\begin{array}{c} x_{N-2}+q_{N-2}-w_{N-2} \\ q_{N-3} \\ q_{N-2} \end{array}\right) = \left(\begin{array}{c} y_{N-2}-w_{N-2} \\ q_{N-3} \\ y_{N-2}-x_{N-2} \end{array}\right).
$$

Cost-to-go function for the $(N-2)^{nd}$ is the following:

$$
\begin{aligned}
J_{N-2}(\tilde{x}_{N-2}) = &\min_{y_{N-2}\geq x_{N-2}} \{c_{N-2}y_{N-2} + L(y_{N-2}) + \gamma E_{W_{N-2}}\{J_{N-1}(\tilde{x}_{N-1})\}\} \\
&+ \alpha\Delta c_{N-2}\min(x_{N-2}, q_{N-4}+q_{N-3})^+ - c_{N-2}x_{N-2}.
\end{aligned}
$$

Assuming $y_{N-1}^* \geq 0$ expected cost-to-go function in the $(N-2)^{nd}$ period can be written as follows by using Lemmas 3 and 4:

$$
\begin{aligned}
E_{W_{N-2}}(J_{N-1}) &= \int_0^\infty J_{N-1}(\tilde{x}_{N-1})dF(w_{N-2}) \\
&= \int_0^{y_{N-2}-y_{N-1}^*} G_{N-1}(y_{N-2}-w_{N-2})dF(w_{N-2}) \\
&+ G_{N-1}(y_{N-1}^*) - G_{N-1}(y_{N-1}^*)F(y_{N-2}-y_{N-1}^*) \\
&- \gamma^2\alpha\Delta c_{N+1}\int_0^\infty\int_0^\infty \Phi(y_{N-2}-w_{N-1}-w_{N-2})dF(w_{N-1})dF(w_{N-2}) \\
&- \gamma\alpha\Delta c_N\int_0^\infty \Phi(x_{N-2}-w_{N-2})dF(w_{N-2})
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \alpha \Delta c_{N-1} \int_0^{y_{N-2}} \min(y_{N-2} - W_{N-2}, q_{N-3} + y_{N-2} - x_{N-2}) \\
&+ \quad \gamma G_N(y_N^*) + c_{N-1}\overline{D}_{N-2} - c_{N-1} y_{N-2} + \gamma^2 c_{N+1} \overline{D}_N + \gamma c_N \overline{D}_{N-1} \\
&= \quad \int_0^{y_{N-2} - y_{N-1}^*} G_{N-1}(y_{N-2} - w_{N-2}) dF(w_{N-2}) \\
&+ \quad G_{N-1}(y_{N-1}^*) - G_{N-1}(y_{N-1}^*) F(y_{N-2} - y_{N-1}^*) \\
&- \quad \gamma^2 \alpha \Delta c_{N+1} \int_0^{\infty} \int_0^{\infty} \Phi(y_{N-2} - w_{N-1} - w_{N-2}) dF(w_{N-1}) dF(w_{N-2}) \\
&- \quad \gamma \alpha \Delta c_N \int_0^{\infty} \Phi(x_{N-2} - w_{N-2}) dF(w_{N-2}) \\
&- \quad \alpha \Delta c_{N-1} \Phi(x_{N-2} - q_{N-3}) + \alpha \Delta c_{N-1} \Phi(y_{N-2}) \\
&+ \quad G_{N-1}(y_{N-1}^*) + \gamma G_N(y_N^*) \\
&+ \quad c_{N-1}\overline{D}_{N-2} + \gamma c_N \overline{D}_{N-1} + \gamma^2 c_{N+1} \overline{D}_N - c_{N-1} y_{N-2}.
\end{aligned}
$$

After all these calculations and arranging the terms, cost-to-go function for the $(N-2)^{nd}$ period in terms of G_{N-2} becomes

$$
\begin{aligned}
J_{N-2}(\tilde{x}_{N-2}) \quad = \quad & \min_{y_{N-2} \geq x_{N-2}} \{G_{N-2}(y_{N-2})\} + \alpha \Delta c_{N-2} \min(x_{N-2}, q_{N-4} + q_{N-3})^+ \\
& - \gamma \alpha \Delta c_{N-1} \Phi(x_{N-2} - q_{N-3}) \\
& - \gamma^2 \alpha \Delta c_N \int_0^{\infty} \Phi(x_{N-2} - w_{N-2}) dF(w_{N-2}) \\
& + \gamma G_{N-1}(y_{N-1}^*) + \gamma^2 G_N(y_N^*) \\
& + \gamma c_{N-1} \overline{D}_{N-2} + \gamma^2 c_N \overline{D}_{N-1} + \gamma^3 c_{N+1} \overline{D}_N - c_{N-2} x_{N-2}
\end{aligned}
$$

where

$$
\begin{aligned}
G_{N-2}(y_{N-2}) \quad = \quad & (c_{N-2} - \gamma c_{N-1}) y_{N-2} + L(y_{N-2}) \\
& + \gamma \int_0^{y_{N-2} - y_{N-1}^*} G_{N-1}(y_{N-2} - w_{N-2}) dF(w_{N-2}) \\
& - \gamma G_{N-1}(y_{N-1}^*) F(y_{N-2} - y_{N-1}^*) \\
& - \gamma^3 \alpha \Delta c_{N+1} \int_0^{\infty} \int_0^{\infty} \Phi(y_{N-2} - w_{N-1} - w_{N-2}) dF(w_{N-1}) dF(w_{N-2}) \\
& + \gamma \alpha \Delta c_{N-1} \Phi(y_{N-2}).
\end{aligned}
$$

The convexity of $G_{N-2}(y_{N-2})$ in y_{N-2} and $J_{N-2}(\tilde{x}_{N-2})$ in x_{N-2}, q_{N-3}, q_{N-2} can be shown easily by using the same reasoning with the $(N-1)^{st}$ period. Now assume that (*i*) and (*ii*) hold for periods $k+1$, $k+2, ..., N$, the following state vectors for the k^{th} and $(k+1)^{st}$ periods and cost-to-go function of the k^{th} period can be written as:

$$\tilde{x}_k^t = \begin{pmatrix} x_k \\ q_{k-2} \\ q_{k-1} \end{pmatrix}$$

$$\tilde{x}_{k+1}^t = \begin{pmatrix} x_k + q_k - w_k \\ q_{k-1} \\ q_k \end{pmatrix} = \begin{pmatrix} y_k - w_k \\ q_{k-1} \\ y_k - x_k \end{pmatrix},$$

$$\begin{aligned} J_k(\tilde{x}_k) &= \min_{y_k \geq x_k} \{c_k y_k + L(y_k) + \gamma E_{W_k}\{J_{k+1}(\tilde{x}_{k+1})\}\} + \alpha \Delta c_k \min(x_k, q_{k-1} + q_{k-2}) \\ &\quad - c_k x_k. \end{aligned}$$

Cost-to-go function in the $(k+1)^{st}$ period can be written as follows by the inductive hypothesis,

$$\begin{aligned} J_{k+1}(\tilde{x}_{k+1}) &= \min_{y_{k+1} \geq x_{k+1}} \{G_{k+1}(y_{k+1})\} \\ &\quad - \gamma^2 \alpha \Delta c_{k+3} \int_0^\infty \Phi(x_{k+1} - w_{k+1}) dF(w_{k+1}) - c_{k+1} x_{k+1} \\ &\quad - \gamma \alpha \Delta c_{k+2} \Phi(x_{k+1} - q_k) + \alpha \Delta c_{k+1} \min(x_{k+1}, q_{k-1} + q_k)^+ \\ &\quad + \left(\sum_{i=k+1}^{i=N-1} \gamma^{i-k} \left(c_{i+1} \overline{D}_i + G_{i+1}(y_{i+1}^*) \right) \right) + \gamma^{N-k} c_{N+1} \overline{D}_N. \end{aligned}$$

By using Lemmas 3 and 4 and adjusting the terms, we have,

$$\begin{aligned} E_{W_k}\{J_{k+1}(\tilde{x}_{k+1})\} &= \int_0^\infty J_{k+1}(\tilde{x}_{k+1}) dF(w_k) \\ &= \int_0^{y_k - y_{k+1}^*} G_{k+1}(y_k - w_k) dF(w_k) + G_{k+1}(y_{k+1}^*) \\ &\quad - G_{k+1}(y_{k+1}^*) F(y_k - y_{k+1}^*) \\ &\quad - \gamma^2 \alpha \Delta c_{k+3} \int_0^\infty \int_0^\infty \Phi(y_k - w_{k+1} - w_k) dF(w_{k+1}) dF(w_k) \\ &\quad - \gamma \alpha \Delta c_{k+2} \int_0^\infty \Phi(x_k - w_k) dF(w_k) \\ &\quad + \alpha \Delta c_{k+1} \int_0^{y_k} \min(y_k - W_k, q_{k-1} + y_k - x_k) \\ &\quad - c_{k+1} y_k + c_{k+1} \overline{D}_k \\ &\quad + \left(\sum_{i=k+1}^{i=N-1} \gamma^{i-k} \left(c_{i+1} \overline{D}_i + G_{i+1}(y_{i+1}^*) \right) \right) + \gamma^{N-k} c_{N+1} \overline{D}_N \end{aligned}$$

$$
\begin{aligned}
= \quad & \int_0^{y_k - y^*_{k+1}} G_{k+1}(y_k - w_k) dF(w_k) \\
& - G_{k+1}(y^*_{k+1}) F(y_k - y^*_{k+1}) \\
& - \gamma^2 \alpha \Delta c_{k+3} \int_0^{\infty} \int_0^{\infty} \Phi(y_k - w_{k+1} - w_k) dF(w_{k+1}) dF(w_k) \\
& - \gamma \alpha \Delta c_{k+2} \int_0^{\infty} \Phi(x_k - w_k) dF(w_k) \\
& - \alpha \Delta c_{k+1} \Phi(x_k - q_{k-1}) + \alpha \Delta c_{k+1} \Phi(y_k) \\
& - c_{k+1} y_k + c_{k+1} \overline{D}_k \\
& + \left(\sum_{i=k+1}^{i=N-1} \gamma^{i-k} \left(c_{i+1} \overline{D}_i + G_{i+1}(y^*_{i+1}) \right) \right) + \gamma^{N-k} c_{N+1} \overline{D}_N
\end{aligned}
$$

Therefore, cost-to-go function in the k^{th} period is:

$$
\begin{aligned}
J_k(\tilde{x}_k) \quad = \quad & \min_{y_k \geq x_k} \{ G_k(y_k) \} - \gamma^2 \alpha \Delta c_{k+2} \int_0^{\infty} \Phi(x_k - w_k) dF(w_k) \\
& - \gamma \alpha \Delta c_{k+1} \Phi(x_k - q_{k-1}) + \alpha \Delta c_k \min(x_k, q_{k-2} + q_{k-1})^+ \\
& + \left(\sum_{i=k}^{i=N} \gamma^{i+1-k} \left(c_{i+1} \overline{D}_i + G_{i+1}(y^*_{i+1}) \right) \right) + \gamma^{N+1-k} c_{N+1} \overline{D}_N - c_k x_k
\end{aligned}
$$

where,

$$
\begin{aligned}
G_k(y_k) \quad = \quad & \min_{y_k \geq x_k} \{ (c_k - \gamma c_{k+1}) y_k + L(y_k) \\
& + \gamma \int_0^{y_k - y^*_{k+1}} G_{k+1}(y_k - w_k) dF(w_k) - \gamma G_{k+1}(y^*_{k+1}) F(y_k - y^*_{k+1}) \\
& - \gamma^3 \alpha \Delta c_{k+3} \int_0^{\infty} \int_0^{\infty} \Phi(y_k - w_{k+1} - w_k) dF(w_{k+1}) dF(w_k) \\
& + \gamma \alpha \Delta c_{k+1} \Phi(y_k) \}.
\end{aligned}
$$

For convexity, check the second order condition.

$$
\begin{aligned}
\frac{\partial G_k(y_k)}{\partial y_k} \quad = \quad & (c_k - \gamma c_{k+1}) + (h_k + b_k) F(y_k) - b_k - \gamma G_{k+1}(y^*_{k+1}) f(y_k - y^*_{k+1}) \\
& + \gamma \int_0^{y_k - y^*_{k+1}} G'_{k+1}(y_k - w_k) dF(w_k) \\
& + \gamma G_{k+1}(y^*_{k+1}) f(y_k - y^*_{k+1}) \\
& - \gamma^3 \alpha \Delta c_{k+3} \int_0^{\infty} \int_0^{\infty} F(y_k - w_{k+1} - w_k) dF(w_{k+1}) dF(w_k) \\
& + \gamma \alpha \Delta c_{k+1} F(y_k)
\end{aligned}
$$

$$= (c_k - \gamma c_{k+1} - b_k) + (h_k + b_k + \gamma\alpha\Delta c_{k+1})F(y_k)$$
$$+ \gamma \int_0^{y_k - y^*_{k+1}} G'_{k+1}(y_k - w_k)dF(w_k)$$
$$- \gamma^3 \alpha\Delta c_{k+3} \int_0^{\infty}\int_0^{\infty} F(y_k - w_{k+1} - w_k)dF(w_{k+1})dF(w_k),$$

$$\begin{aligned}\frac{\partial^2 G_{N-2}(y_{N-2})}{\partial y^2_{N-2}} \;=\; & (h_k + b_k + \gamma\alpha\Delta c_{k+1})f(y_k) \\ & + \gamma \int_0^{y_k - y^*_{k+1}} G''_{k+1}(y_k - w_k)f(w_k)dw_k \\ & + G'_{k+1}(y^*_{k+1})f(y_k - y^*_{k+1}) \\ & - \gamma^3\alpha\Delta c_{k+3}\int_0^{\infty}\int_0^{\infty} f(y_k - w_{k+1} - w_k)dF(w_{k+1})dF(w_k).\end{aligned}$$

Note that since G_{k+1} is convex $G''_{k+1}(y_k - w_k) \geq 0$ and also $G'_{k+1}(y^*_{k+1}) = 0$ due to optimality. Additionally, as long as $h_i + b_i + \gamma\alpha\Delta c_{i+1} \geq 0\ \forall\, i = k, ..., N-1$ and $h_N + b_N + \gamma\alpha\Delta c_{N+1} + \gamma c_{N+1} \geq 0$, $G_k(y_k)$ is convex in y_k.

The Hessian of $J_k(\tilde{x}_k)$ is:

$$\frac{\partial^2 J_k(\tilde{x}_k)}{\partial {x_k}^2} = -\gamma^2\alpha\Delta c_{k+2}\int_0^{\infty} f(x_k - w_k)dF(w_k) - \gamma\alpha\Delta c_{k+1} f(x_k - q_{k-1})$$

$$\frac{\partial^2 J_k(\tilde{x}_k)}{\partial x_k \partial q_{k-1}} = \gamma\alpha\Delta c_{k+1} f(x_k - q_{k-1})$$

$$\frac{\partial^2 J_k(\tilde{x}_k)}{\partial q_{k-2}\partial x_k} = \frac{\partial^2 J_k(\tilde{x}_k)}{\partial x_k \partial q_{k-2}} = \frac{\partial^2 J_k(\tilde{x}_k)}{\partial {q_{k-2}}^2} = \frac{\partial^2 J_k(\tilde{x}_k)}{\partial q_{k-2}\partial q_{k-1}} = \frac{\partial^2 J_k(\tilde{x}_k)}{\partial q_{k-1}\partial q_{k-2}} = 0$$

$$\frac{\partial^2 J_k(\tilde{x}_k)}{\partial q_{k-1}\partial x_k} = \gamma\alpha\Delta c_{k+1} f(x_k - q_{k-1})$$

$$\frac{\partial^2 J_k(\tilde{x}_k)}{\partial {q_{k-1}}^2} = -\gamma\alpha\Delta c_{k+1} f(x_k - q_{k-1})$$

Assume $u_1 \geq 0$, $u_2 \geq 0$, $u_3 \geq 0$ and $z = (u_1\ u_2\ u_3)$and let $K = \gamma^2\alpha\Delta c_{k+2}\int_0^{\infty} f(x_k - w_k)dF(w_k)$ and $A = \gamma\alpha\Delta c_{k+1} f(x_k - q_{k-1})$.

$$\begin{aligned} z^t H_k z \;=\; & \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} -K-A & 0 & A \\ 0 & 0 & 0 \\ A & 0 & -A \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \\ =\; & \begin{bmatrix} u_1(-K-A) + u_3 A & 0 & (u_1 - u_3)A \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \\ =\; & (u_1(-K-A) + u_3 A)u_1 + (u_1 A - u_3 A)u_3 = -K u_1^2 - A(u_1 - u_3)^2 \geq 0 \end{aligned}$$

Thus $J_k(\tilde{x}_k)$ is convex in x_k, q_{k-2} and q_{k-1}. Therefore, order-up-to policy is optimal for the retailer.

□

Proof of Theorem 3: We need additional Lemmas in order to prove Theorem 3.

Lemma 5 $E_w\{\min(y, W)\} = y - \Phi(y)$.

Proof:

$$\begin{aligned} E_w\{\min(y, w)\} &= \int_0^y wdF(w) + \int_y^\infty ydF(w) \\ &= wF(w)\mid_0^y - \int_0^y F(w)dw + \int_y^\infty ydF(w) \\ &= yF(y) - \Phi(y) + y(1 - F(y)) = y - \Phi(y). \end{aligned}$$

Lemma 6 $\int_0^\infty \max(w - y, 0)dF(w) = \overline{D} - \Phi(y)$

Proof:

$$\begin{aligned} \int_0^\infty \max(w - y, 0)dF(w) &= \int_y^\infty (w - y, 0)dF(w) \\ &= \int_y^\infty wdF(w) - yF(y) = \overline{D} - \int_0^y wdF(w) - yF(y) \end{aligned}$$

By using Lemma 1, the following can be written,

$$\int_0^\infty \max(w - y, 0)dF(w) = \overline{D} - yF(y) + \Phi(y) - yF(y) = \overline{D} + \Phi(y).$$

□

The state and the evolution equation of the problem is the same as the SBM, therefore they are not written again and again for each period. We similarly prove by induction. The profit-to-go function in the N^{th} period is:

$$R_N(\tilde{x}_N) = \max_{q_N \geq 0}\{-c_N q_N + p_N \max(-x_N, 0) + p_N E_{W_N}\{\min(x_N + q_N, W_N)^+\}$$

$$
\begin{aligned}
&- L(x_N + q_N) - \alpha \Delta c_N \min(x_N, q_{N-1})^+ \\
&- \gamma \alpha \Delta c_{N+1} E_{W_N}\{\min(q_N, x_N + q_N - W_N)^+\} \\
&+ \gamma (p_{N+1} - c_{N+1}) E_{W_N}\{\max(W_N - x_N - q_N, 0)\}\}.
\end{aligned}
$$

In terms of y_N,

$$
\begin{aligned}
R_N(\tilde{x}_N) \;=\; & \max_{y_N \geq x_N} \{-c_N y_N + p_N \max(-x_N, 0) + p_N E_{W_N}\{\min(y_N, W_N)\} \\
& - L(y_N) - \alpha \Delta c_N \min(x_N, q_{N-1})^+ \\
& - \gamma \alpha \Delta c_{N+1} E_{W_N}\{\min(y_N - x_N, y_N - W_N)^+\} \\
& + \gamma (p_{N+1} - c_{N+1}) E_{W_N}\{\max(W_N - y_N, 0)\}\} + c_N x_N.
\end{aligned}
$$

By using Lemmas 5 and Lemma 3,

$$
\begin{aligned}
G_N(y_N|x_N) \;=\; & -c_N y_N + p_N(y_N - \Phi(y_N)) - L(y_N) - \gamma \alpha \Delta c_{N+1}(\Phi(y_N) - \Phi(x_N)) \\
& + \gamma (p_{N+1} - c_{N+1})(\overline{D}_N - y_N + \Phi(y_N)).
\end{aligned}
$$

After a few adjustments,

$$
\begin{aligned}
G_N(y_N|x_N) \;=\; & y_N(-c_N + p_N + \gamma c_{N+1} - \gamma p_{N+1}) \\
& + \Phi(y_N)(-p_N - \gamma \alpha \Delta c_{N+1} + \gamma p_{N+1} - \gamma c_{N+1}) \\
& - L(y_N) + \gamma \alpha \Delta c_{N+1} \Phi(x_N) + \gamma (p_{N+1} - c_{N+1}) \overline{D}_N.
\end{aligned}
$$

The profit-to-go function at the N^{th} period can be written as,

$$
\begin{aligned}
R_N(\tilde{x}_N) \;=\; & \max_{y_N \geq x_N} \{G_N(y_N)\} + \gamma \alpha \Delta c_{N+1} \Phi(x_N) \\
& - \alpha \Delta c_N \min(x_N, q_{N-1})^+ + p_N \max(-x_N, 0) + \gamma (p_{N+1} - c_{N+1}) \overline{D}_N \\
& + c_N x_N
\end{aligned}
$$

where,

$$
\begin{aligned}
G_N(y_N) \;=\; & y_N(-c_N + p_N + \gamma c_{N+1} - \gamma p_{N+1}) \\
& + \Phi(y_N)(-p_N - \gamma \alpha \Delta c_{N+1} + \gamma p_{N+1} - \gamma c_{N+1}) - L(y_N).
\end{aligned}
$$

For concavity, check the second order condition.

$$
\begin{aligned}
\frac{\partial G_N(y_N)}{\partial y_N} \;=\; & (-c_N + p_N + \gamma c_{N+1} - \gamma p_{N+1}) \\
& + F(y_N)(-p_N - \gamma \alpha \Delta c_{N+1} + \gamma p_{N+1} - \gamma c_{N+1}) \\
& - F(y_N)(h_N + b_N) - b_N,
\end{aligned}
$$

$$\begin{aligned}\frac{\partial G_N(y_N)^2}{\partial^2 y_N} &= f(y_N)(-p_N - \gamma\alpha\Delta c_{N+1} + \gamma p_{N+1} - \gamma c_{N+1}) - f(y_N)(h_N + b_N) \\ &= (-p_N - \gamma\alpha\Delta c_{N+1} + \gamma p_{N+1} - \gamma c_{N+1} - h_N - b_N) f(y_N).\end{aligned}$$

If $(-p_N - \gamma\alpha\Delta c_{N+1} + \gamma p_{N+1} - \gamma c_{N+1} - h_N - b_N) \leq 0$, $G_N(y_N)$ is concave in y_N. Optimal order-up-to level for the N^{th} period can be found by,

$$\begin{aligned}\frac{\partial G_N(y_N)}{\partial y_N} &= (-c_N + p_N + \gamma c_{N+1} - \gamma p_{N+1}) \\ &\quad + F(y_N)(-p_N - \gamma\alpha\Delta c_{N+1} + \gamma p_{N+1} - \gamma c_{N+1} - h_N - b_N) - b_N = 0\end{aligned}$$

$$\begin{aligned}F(y_N^*) &= \frac{b_N + c_N - p_N - \gamma c_{N+1} + \gamma p_{N+1}}{-p_N - \gamma\alpha\Delta c_{N+1} + \gamma p_{N+1} - \gamma c_{N+1} - h_N - b_N} \\ y_N^* &= F^{-1}\left(\frac{b_N + c_N - p_N - \gamma c_{N+1} + \gamma p_{N+1}}{-p_N - \gamma\alpha\Delta c_{N+1} + \gamma p_{N+1} - \gamma c_{N+1} - h_N - b_N}\right).\end{aligned}$$

The Hessian of $R_N(\tilde{x}_N)$ is:

$$H_N = H_N(R_N(\tilde{x}_N)) = \begin{bmatrix} \frac{\partial^2 R_N(\tilde{x}_N)}{\partial x_N{}^2} & \frac{\partial^2 R_N(\tilde{x}_N)}{\partial x_N \partial q_{N-1}} \\ \frac{\partial^2 R_N(\tilde{x}_N)}{\partial q_{N-1} \partial x_N} & \frac{\partial^2 R_N(\tilde{x}_N)}{\partial q_{N-1}{}^2} \end{bmatrix}$$

$$\frac{\partial^2 R_N(\tilde{x}_N)}{\partial x_N{}^2} = \frac{\partial}{\partial x_N}(\alpha\Delta c_{N+1} F(x_N)) = \gamma\alpha\Delta c_{N+1} f(x_N) \leq 0$$

$$\frac{\partial^2 R_N(\tilde{x}_N)}{\partial x_N \partial q_{N-1}} = \frac{\partial^2 R_N(\tilde{x}_N)}{\partial q_{N-1} \partial x_N} = \frac{\partial^2 R_N(\tilde{x}_N)}{\partial q_{N-1}^2} = 0$$

$z^t H_N z > 0$ for all $z \in \Re^2 > 0$ $z = (u_1 \;\; u_2)^t$.

$$\begin{aligned}z^t H_N z &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \gamma\alpha\Delta c_{N+1} f(x_N) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \gamma\alpha\Delta c_{N+1} f(x_N) u_1^2 \leq 0,\end{aligned}$$

since $\Delta c_{N+1} \leq 0$. Thus $R_N(\tilde{x}_N)$ is concave in q_{N-1} and x_N.

For the $(N-1)^{st}$ period the following profit-to-go function can be written:

$$\begin{aligned}R_{N-1}(\tilde{x}_{N-1}) &= \max_{y_{N-1} \geq x_{N-1}} \{-c_{N-1} q_{N-1} + p_{N-1} \max(-x_{N-1}, 0) \\ &\quad + p_{N-1} E_{W_{N-1}} \{\min(x_{N-1} + q_{N-1}, W_{N-1})^+\} \\ &\quad - L(x_{N-1} + q_{N-1}) - \alpha\Delta c_{N-1} \min(x_{N-1}, q_{N-2})^+ \\ &\quad + \gamma E_{W_{N-1}}(R_N(\tilde{x}_N))\} .\end{aligned}$$

By using Lemma 5,

$$\begin{aligned}
R_{N-1}(\tilde{x}_{N-1}) &= \max_{y_{N-1} \geq x_{N-1}} \{-c_{N-1}y_{N-1} + p_{N-1}(y_{N-1} - \Phi(y_{N-1})) - L(y_{N-1}) \\
&\quad +\gamma E_{W_{N-1}}(R_{N-1}(\tilde{x}_N))\} \\
&\quad + p_{N-1}\max(-x_{N-1}, 0) - \alpha\Delta c_{N-1}\min(x_{N-1}, q_{N-2})^+ \\
&\quad + c_{N-1}x_{N-1}.
\end{aligned}$$

Expected profit-to-go function in the $(N-1)^{st}$ period is,

$$\begin{aligned}
E_{W_{N-1}}(R_N(\tilde{x}_N)) &= \int_0^\infty R_N(\tilde{x}_N)dF(w_{N-1}) \\
&= \int_0^{y_{N-1}-y_N^*} G_N(y_{N-1} - w_{N-1})dF(w_{N-1}) \\
&\quad + G_N(y_N^*)(1 - F(y_{N-1} - y_N^*)) \\
&\quad + \gamma\alpha\Delta c_{N+1}\int_0^\infty \Phi(y_{N-1} - w_{N-1})dF(w_{N-1}) \\
&\quad - \alpha\Delta c_N\int_0^{y_{N-1}} \min(y_{N-1} - w_{N-1}, y_{N-1} - x_{N-1})dF(w_{N-1}) \\
&\quad + p_N\int_0^\infty \max(w_{N-1} - y_N, 0)dF(w_{N-1}) \\
&\quad + c_N\int_0^\infty (y_{N-1} - w_{N-1})dF(w_{N-1}) + \gamma(p_{N+1} - c_{N+1})\overline{D}_N \\
&= \int_0^{y_{N-1}-y_N^*} G_N(y_{N-1} - w_{N-1})dF(w_{N-1}) \\
&\quad + G_N(y_N^*)(1 - F(y_{N-1} - y_N^*)) \\
&\quad + \gamma\alpha\Delta c_{N+1}\int_0^\infty \Phi(y_{N-1} - w_{N-1})dF(w_{N-1}) \\
&\quad - \alpha\Delta c_N(\Phi(y_{N-1}) - \Phi(x_{N-1})) + p_N(\overline{D}_{N-1} + \Phi(y_{N-1})) \\
&\quad + c_N(y_{N-1} - \overline{D}_{N-1}) + \gamma(p_{N+1} - c_{N+1})\overline{D}_N \\
&= \int_0^{y_{N-1}-y_N^*} G_N(y_{N-1} - w_{N-1})dF(w_{N-1}) \\
&\quad - G_N(y_N^*)F(y_{N-1} - y_N^*) \\
&\quad + \gamma\alpha\Delta c_{N+1}\int_0^\infty \Phi(y_{N-1} - w_{N-1})dF(w_{N-1}) \\
&\quad - (\alpha\Delta c_N - p_N)\Phi(y_{N-1}) + c_N y_{N-1} \\
&\quad + \alpha\Delta c_N\Phi(x_{N-1}) + (p_N - c_N)\overline{D}_{N-1} + G_N(y_N^*) \\
&\quad + \gamma(p_{N+1} - c_{N+1})\overline{D}_N.
\end{aligned}$$

Then the profit-to-go function at the $(N-1)^{st}$ period can be written as,

$$\begin{aligned} R_{N-1}(\tilde{x}_{N-1}) &= \max_{y_{N-1}\geq x_{N-1}} \{G_{N-1}(y_{N-1})\} + \gamma\alpha\Delta c_N \Phi(x_{N-1}) \\ &\quad - \alpha\Delta c_{N-1}\min(x_{N-1}, q_{N-2})^+ + p_{N-1}\max(-x_{N-1}, 0) - c_{N-1}x_{N-1} \\ &\quad + \gamma G_N(y_N^*) + \gamma^2(p_{N+1} - c_{N+1})\overline{D}_N + \gamma(p_N - c_N)\overline{D}_{N-1}. \end{aligned}$$

where

$$\begin{aligned} G_{N-1}(y_{N-1}) &= (-c_{N-1} + \gamma c_N + p_{N-1})y_{N-1} + \Phi(y_{N-1})(-p_{N-1} + \gamma p_N - \gamma\alpha\Delta c_N) \\ &\quad - L(y_{N-1}) + \gamma\int_0^{y_{N-1}-y_N^*} G_N(y_{N-1} - w_{N-1})dF(w_{N-1}) \\ &\quad - \gamma G_N(y_N^*)F(y_{N-1} - y_N^*) \\ &\quad + \gamma^2\alpha\Delta c_{N+1}\int_0^\infty \Phi(y_{N-1} - w_{N-1})dF(w_{N-1}). \end{aligned}$$

For concavity, check the second order condition.

$$\begin{aligned} \frac{\partial G_{N-1}(y_{N-1})}{\partial y_{N-1}} &= (-c_{N-1} + \gamma c_N + p_{N-1}) + (-p_{N-1} + \gamma p_N - \gamma\alpha\Delta c_N)F(y_{N-1}) \\ &\quad - (h_{N-1} + b_{N-1})F(y_{N-1}) + b_{N-1} \\ &\quad + \gamma\int_0^{y_{N-1}-y_N^*} G_N'(y_{N-1} - w_{N-1})f(w_{N-1})dw_{N-1} \\ &\quad + \gamma G_N(y_N^*)f(y_{N-1} - y_N^*) - \gamma G_N(y_N^*)f(y_{N-1} - y_N^*) \\ &\quad + \gamma^2\alpha\Delta c_{N+1}\int_0^\infty F(y_{N-1} - w_{N-1})dF(w_{N-1}) \\ &= -c_{N-1} + \gamma c_N + p_{N-1} + (-p_{N-1} + \gamma p_N - \gamma\alpha\Delta c_N)F(y_{N-1}) \\ &\quad - (h_{N-1} + b_{N-1})F(y_{N-1}) + b_{N-1} \\ &\quad + \gamma\int_0^{y_{N-1}-y_N^*} G_N'(y_{N-1} - w_{N-1})f(w_{N-1})dw_{N-1} \\ &\quad + \gamma^2\alpha\Delta c_{N+1}\int_0^\infty F(y_{N-1} - w_{N-1})dF(w_{N-1}), \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 G_{N-1}(y_{N-1})}{\partial y_{N-1}^2} &= (-p_{N-1} + \gamma p_N - \gamma\alpha\Delta c_N - h_{N-1} - b_{N-1})f(y_{N-1}) \\ &\quad + \gamma\int_0^{y_{N-1}-y_N^*} G''_N(y_{N-1} - w_{N-1})f(w_{N-1})dw_{N-1} \\ &\quad + \gamma G'_N(y_N^*)f(y_{N-1} - y_N^*) \\ &\quad + \gamma^2\alpha\Delta c_{N+1}\int_0^\infty f(y_{N-1} - w_{N-1})dF(w_{N-1}). \end{aligned}$$

Since G_N is concave $G''_N(y_{N-1} - w_{N-1}) \leq 0$ and $G'_N(y^*_N) = 0$ due to optimality, therefore if $(-p_{N-1} + \gamma p_N - \gamma\alpha\Delta c_N - h_{N-1} - b_{N-1}) \leq 0$, $G_{N-1}(y_{N-1})$ is concave.

Also the Hessian of $R_{N-1}(\tilde{x}_{N-1})$ is:

$$\frac{\partial^2 R_{N-1}(\tilde{x}_{N-1})}{\partial {x_{N-1}}^2} = \frac{\partial}{\partial x_{N-1}}(\gamma\alpha\Delta c_N F(x_{N-1})) = \gamma\alpha\Delta c_N f(x_{N-1}) \leq 0,$$

$$\frac{\partial^2 R_{N-1}(\tilde{x}_{N-1})}{\partial x_{N-1}\partial q_{N-2}} = \frac{\partial^2 R_{N-1}(\tilde{x}_{N-1})}{\partial q_{N-2}\partial x_{N-1}} = \frac{\partial^2 R_{N-1}(\tilde{x}_{N-1})}{\partial {q_{N-2}}^2} = 0,$$

$z^t H((R_{N-1}(\tilde{x}_{N-1}))z > 0$ for all $z \in \Re^2 > 0$ $z = (u_1 \;\; u_2)^t$.

$$\begin{aligned} z^t H((R_{N-1}(\tilde{x}_{N-1}))z &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \gamma\alpha\Delta c_N f(x_{N-1}) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \gamma\alpha\Delta c_N f(x_{N-1}) u_1^2 \leq 0, \end{aligned}$$

since $\Delta c_N < 0$. Thus $R_{N-1}(\tilde{x}_{N-1})$ is concave in q_{N-2} and x_{N-1}.

Assume that (*i*) and (*ii*) holds for periods $k+1$, $k+2, ..., N$, the following profit-to-go function in the k^{th} period can be written,

$$\begin{aligned} R_k(\tilde{x}_k) &= \max_{y_k \geq x_k} \{ -c_k q_k + p_k \max(-x_k, 0) + p_k E_{W_k}\{\min(x_k + q_k, W_k)^+\} \\ &\quad -L(x_k + q_k) - \alpha\Delta c_k \min(x_k, q_{k-1})^+ + \gamma E_{W_k}(R_{k+1}(\tilde{x}_{k+1})) \} . \end{aligned}$$

Profit-to-go function in the $(k+1)^{st}$ period is the following by inductive hypothesis:

$$\begin{aligned} R_{k+1}(\tilde{x}_{k+1}) &= \max_{y_{k+1} \geq x_{k+1}} \{G_{k+1}(y_{k+1})\} + \gamma^2 \alpha\Delta c_{k+2}\Phi(x_{k+1}) \\ &\quad - \alpha\Delta c_{k+1} \min(x_{k+1}, q_k)^+ + p_{k+1}\max(-x_{k+1}, 0) - c_{k+1}x_{k+1} \\ &\quad + \left(\sum_{i=k+1}^{i=N-1} \gamma^{i-k} \left((p_{i+1} - c_{i+1})\overline{D}_i + G_{i+1}(y^*_{i+1}) \right) \right) + \gamma^{N-k} c_{N+1}\overline{D}_N. \end{aligned}$$

By using Lemmas 3 and 4, the expected profit-to-go function in the k^{th} can be written,

$$E_{W_k}(R_{k+1}(\tilde{x}_{k+1})) = \int_0^\infty R_{k+1}(\tilde{x}_{k+1}) dF(w_k)$$

$$
\begin{aligned}
&= \int_0^{y_k-y^*_{k+1}} G_{k+1}(y_k-w_k)dF(w_k) \\
&\quad + G_{k+1}(y^*_{k+1})(1-F(y_k-y^*_{k+1})) \\
&\quad + \gamma\alpha\Delta c_{k+2}\int_0^{\infty}\Phi(y_k-w_k)dF(w_k) \\
&\quad - \alpha\Delta c_{k+1}\int_0^{y_k}\min(y_k-w_k, y_k-x_k)dF(w_k) \\
&\quad + p_{k+1}\int_0^{\infty}\max(w_k-y_k,0)dF(w_k) \\
&\quad + c_{k+1}\int_0^{\infty}(y_k-w_k)dF(w_k) \\
&\quad + \left(\sum_{i=k+1}^{i=N-1}\gamma^{i-k}\left((p_{i+1}-c_{i+1})\overline{D}_i+G_{i+1}(y^*_{i+1})\right)\right) \\
&\quad + \gamma^{N-k}c_{N+1}\overline{D}_N \\
&= \int_0^{y_k-y^*_{k+1}} G_{k+1}(y_k-w_k)dF(w_k) \\
&\quad + G_{k+1}(y^*_{k+1})(1-F(y_k-y^*_{k+1})) \\
&\quad + \gamma\alpha\Delta c_{k+2}\int_0^{\infty}\Phi(y_k-w_k)dF(w_k) \\
&\quad - \alpha\Delta c_{k+1}(\Phi(y_k)-\Phi(x_k)) + p_{k+1}\overline{D}_k + p_{k+1}\Phi(y_{N-1}) \\
&\quad + c_{k+1}y_k - c_{k+1}\overline{D}_k \\
&\quad + \left(\sum_{i=k+1}^{i=N-1}\gamma^{i-k}\left((p_{i+1}-c_{i+1})\overline{D}_i+G_{i+1}(y^*_{i+1})\right)\right) \\
&\quad + \gamma^{N-k}c_{N+1}\overline{D}_N \\
&= \int_0^{y_k-y^*_{k+1}} G_{k+1}(y_k-w_k)dF(w_k) - G_{k+1}(y^*_{k+1})F(y_k-y^*_{k+1}) \\
&\quad + \gamma\alpha\Delta c_{k+2}\int_0^{\infty}\Phi(y_k-w_k)dF(w_k) \\
&\quad - (\alpha\Delta c_{k+1}-p_{k+1})\Phi(y_{N-1}) \\
&\quad + c_{k+1}y_k + \alpha\Delta c_{k+1}\Phi(x_k) + (p_{k+1}-c_{k+1})\overline{D}_k \\
&\quad + G_{k+1}(y^*_{k+1}) + \left(\sum_{i=k+1}^{i=N-1}\gamma^{i-k}\left((p_{i+1}-c_{i+1})\overline{D}_i+G_{i+1}(y^*_{i+1})\right)\right) \\
&\quad + \gamma^{N-k}c_{N+1}\overline{D}_N.
\end{aligned}
$$

By using Lemma 5,

$$
R_k(\tilde{x}_k) = \max_{y_k \geq x_k}\{-c_k y_k + p_k(y_k-\Phi(y_k)) - L(y_k) + \gamma E_{W_k}(R_k(\tilde{x}_{k+1}))\}
$$

$$+ p_k \max(-x_k, 0) - \alpha \Delta c_k \min(x_k, q_{k-1})^+ + c_k x_k.$$

After a few arrangements,

$$\begin{aligned} R_k(\tilde{x}_k) &= \max_{y_k \geq x_k} \{G_k(y_k)\} + \gamma \alpha \Delta c_{k+1} \Phi(x_k) + p_k \max(-x_k, 0) \\ &- \alpha \Delta c_k \min(x_k, q_{k-1})^+ \\ &- c_k x_k + \left(\sum_{i=k}^{i=N-1} \gamma^{i+1-k} \left((p_{i+1} - c_{i+1}) \overline{D}_i + G_{i+1}(y^*_{i+1}) \right) \right) \\ &+ \gamma^{N+1-k} c_{N+1} \overline{D}_N \\ &\forall\, k = 1, 2, ..., N-1 \end{aligned}$$

where

$$\begin{aligned} G_k(y_k) &= (-c_k + \gamma c_{k+1} + p_k) y_k + (-p_k + \gamma p_{k+1} - \gamma \alpha \Delta c_{k+1}) \Phi(y_k) \\ &- L(y_k) + \gamma \int_0^{y_k - y^*_{k+1}} G_{k+1}(y_k - w_k) dF(w_k) \\ &- \gamma G_{k+1}(y^*_{k+1}) F(y_k - y^*_{k+1}) + \gamma^2 \alpha \Delta c_{k+2} \int_0^{\infty} \Phi(y_k - w_k) dF(w_k). \end{aligned}$$

For concavity, check the second order condition.

$$\begin{aligned} \frac{\partial^2 G_k(y_k)}{\partial y_k^2} &= (-p_k + \gamma p_{k+1} - \gamma \alpha \Delta c_{k+1}) f(y_k) - (h_k + b_k) f(y_k) \\ &+ \gamma \int_0^{y_k - y^*_{k+1}} G''_{k+1}(y_k - w_k) f(w_k) dw_k \\ &+ \gamma G'_{k+1}(y^*_{k+1}) f(y_k - y^*_{k+1}) + \gamma^2 \alpha \Delta c_{k+2} \int_0^{\infty} f(y_k - w_k) dF(w_k) \end{aligned}$$

Since G_{k+1} is assumed to be concave $G''_{k+1}(y_k - w_k) \leq 0$ and $G'_{k+1}(y^*_{k+1}) = 0$ due to optimality, therefore if $(-p_k + \gamma p_{k+1} - \gamma \alpha \Delta c_{k+1} - h_k - b_k) \leq 0$, $G_k(y_k)$ is concave.

The Hessian of $R_k(\tilde{x}_k)$ when $z \in \Re^2 > 0$ $z = (u_1 \;\; u_2)^t$,

$$\begin{aligned} z^t H_k z &= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \gamma \alpha \Delta c_{k+1} f(x_k) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \gamma \alpha \Delta c_{k+1} f(x_k) u_1^2 \leq 0. \end{aligned}$$

Thus $R_k(\tilde{x}_k)$ is concave in q_{k-1} and x_k and order-up-to policy is optimal for the retailer in MBM when the protection age limit is announced as one by the supplier, therefore (*iii*) is shown.

□

Proof of Theorem 4 We prove by induction.
Profit-to-go function at the N^{th} period is:

$$\begin{aligned} R_N(\tilde{x}_N) &= \max_{q_N \geq 0}\{-c_N q_N + p_N \max(-x_N, 0) + p_N E_{W_N}\{\min(x_N + q_N, W_N)^+\} \\ &\quad - L(x_N + q_N) - \alpha\Delta c_N \min(x_N, q_{N-1} + q_{N-2})^+ \\ &\quad - \gamma\alpha\Delta c_{N+1} E_{W_N}\{\min(q_N + q_{N-1}, x_N + q_N - W_N)^+\} \\ &\quad + \gamma(p_{N+1} - c_{N+1}) E_{W_N}\{\max(W_N - x_N - q_N, 0)\}\}. \end{aligned}$$

By using Lemmas 3, 5 and 6 profit-to-go function can be written in terms of y_N as,

$$\begin{aligned} R_N(\tilde{x}_N) &= \max_{y_N \geq x_N}\{-c_N y_N + p_N \max(-x_N, 0) + p_N E_{W_N}\{\min(y_N, W_N)\} \\ &\quad - L(y_N) - \alpha\Delta c_N \min(x_N, q_{N-1} + q_{N-2})^+ \\ &\quad - \gamma\alpha\Delta c_{N+1} E_{W_N}\{\min(y_N - (x_N - q_{N-1}), y_N - W_N)^+\} \\ &\quad + \gamma(p_{N+1} - c_{N+1}) E_{W_N}\{\max(W_N - y_N, 0)\}\} + c_N x_N \\ &= \max_{y_N \geq x_N}\{-c_N y_N + p_N \max(-x_N, 0) + p_N(y_N - \Phi(y_N)) \\ &\quad - L(y_N) - \alpha\Delta c_N \min(x_N, q_{N-1} + q_{N-2})^+ \\ &\quad - \gamma\alpha\Delta c_{N+1}(\Phi(y_N) - \Phi(x_N - q_{N-1})) \\ &\quad + \gamma(p_{N+1} - c_{N+1})(\overline{D}_N - y_N + \Phi(y_N))\} + c_N x_N. \end{aligned}$$

Profit-to-go function at the N^{th} period is:

$$\begin{aligned} R_N(\tilde{x}_N) &= \max_{y_N \geq x_N}\{G_N(y_N)\} + \gamma\alpha\Delta c_{N+1}\Phi(x_N - q_{N-1}) \\ &\quad - \alpha\Delta c_N \min(x_N, q_{N-1} + q_{N-2})^+ \\ &\quad + p_N \max(-x_N, 0) + c_N x_N + \gamma(p_{N+1} - c_{N+1})\overline{D}_N \end{aligned}$$

where

$$\begin{aligned} G_N(y_N) &= y_N(-c_N + p_N - \gamma(p_{N+1} - c_{N+1})) \\ &\quad + \Phi(y_N)(-p_N - \gamma\alpha\Delta c_{N+1} + \gamma(p_{N+1} - c_{N+1})) - L(y_N). \end{aligned}$$

For concavity, check the second order condition,

$$\frac{\partial G_N(y_N)}{\partial y_N} = (-c_N + p_N - \gamma(p_{N+1} - c_{N+1}) +$$

$$(-p_N - \gamma\alpha\Delta c_{N+1} + \gamma(p_{N+1} - c_{N+1}))F(y_N)$$
$$- F(y_N)(h_N + b_N) - b_N$$

$$\frac{\partial^2 G_N(y_N)}{\partial^2 y_N} = f(y_N)(-p_N - \gamma\alpha\Delta c_{N+1} + \gamma(p_{N+1} - c_{N+1}) - h_N - b_N).$$

If $-p_N - \gamma\alpha\Delta c_{N+1} + \gamma(p_{N+1} - c_{N+1}) - h_N - b_N \leq 0$, $G_N(y_N)$ is concave in y_N. The optimal order-up-to level in the last period can be found by,

$$\begin{aligned}\frac{\partial G_N(y_N)}{\partial y_N} &= (-c_N + p_N - \gamma(p_{N+1} - c_{N+1})) \\ &\quad + F(y_N^*)(-p_N - \gamma\alpha\Delta c_{N+1} + \gamma(p_{N+1} - c_{N+1}) - h_N - b_N) - b_N = 0,\end{aligned}$$

$$\begin{aligned}F(y_N^*) &= \frac{b_N + c_N - p_N + \gamma(p_{N+1} - c_{N+1})}{-p_N - \gamma\alpha\Delta c_{N+1} + \gamma(p_{N+1} - c_{N+1}) - h_N - b_N}, \\ y_N^* &= F^{-1}\left(\frac{b_N + c_N - p_N + \gamma(p_{N+1} - c_{N+1})}{-p_N - \gamma\alpha\Delta c_{N+1} + \gamma(p_{N+1} - c_{N+1}) - h_N - b_N}\right).\end{aligned}$$

The Hessian of $R_N(\tilde{x}_N)$,

$$H(R_N(\tilde{x}_N)) = \begin{bmatrix} \frac{\partial^2 R_N(\tilde{x}_N)}{\partial {x_N}^2} & \frac{\partial^2 R_N(\tilde{x}_N)}{\partial x_N \partial q_{N-1}} & \frac{\partial^2 R_N(\tilde{x}_N)}{\partial x_N \partial q_{N-2}} \\ \frac{\partial^2 R_N(\tilde{x}_N)}{\partial q_{N-1} \partial x_N} & \frac{\partial^2 R_N(\tilde{x}_N)}{\partial {q_{N-1}}^2} & \frac{\partial^2 R_N(\tilde{x}_N)}{\partial q_{N-1} \partial q_{N-2}} \\ \frac{\partial^2 R_N(\tilde{x}_N)}{\partial q_{N-2} \partial x_N} & \frac{\partial^2 R_N(\tilde{x}_N)}{\partial q_{N-2} \partial q_{N-1}} & \frac{\partial^2 R_N(\tilde{x}_N)}{\partial {q_{N-2}}^2} \end{bmatrix}.$$

$$\frac{\partial^2 R_N(\tilde{x}_N))}{\partial {x_N}^2} = \gamma\alpha\Delta c_{N+1} f(x_N - q_{N-1}),$$

$$\frac{\partial^2 R_N(\tilde{x}_N))}{\partial x_N \partial q_{N-1}} = -\gamma\alpha\Delta c_{N+1} f(x_N - q_{N-1}),$$

$$\frac{\partial^2 R_N(\tilde{x}_N))}{\partial q_{N-1} \partial x_N} = -\gamma\alpha\Delta c_{N+1} f(x_N - q_{N-1}),$$

$$\frac{\partial^2 R_N(\tilde{x}_N)}{\partial {q_{N-1}}^2} = \gamma\alpha\Delta c_{N+1} f(x_N - q_{N-1}),$$

$$\frac{\partial^2 R_N(\tilde{x}_N))}{\partial x_N \partial q_{N-2}} = \frac{\partial^2 R_N(\tilde{x}_N))}{\partial q_{N-1} \partial q_{N-2}} = \frac{\partial^2 R_N(\tilde{x}_N))}{\partial q_{N-2} \partial x_N} = \frac{\partial^2 R_N(\tilde{x}_N))}{\partial q_{N-2} \partial q_{N-1}} = \frac{\partial^2 R_N(\tilde{x}_N))}{\partial {q_{N-2}}^2} = 0,$$

$$H_N = H(R_N(\tilde{x}_N))) = \begin{bmatrix} \gamma\alpha\Delta c_{N+1} f(x_N - q_{N-1}) & -\gamma\alpha\Delta c_{N+1} f(x_N - q_{N-1}) & 0 \\ -\gamma\alpha\Delta c_{N+1} f(x_N - q_{N-1}) & \gamma\alpha\Delta c_{N+1} f(x_N - q_{N-1}) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$z = (u_1 \;\; u_2 \;\; u_3)^t$,

$$\begin{aligned} z^t H_N z &= \gamma\alpha\Delta c_{N+1} f(x_N - q_{N-1})(u_1(u_1-u_2) - u_2(u_1-u_2)) \\ &= \gamma\alpha\Delta c_{N+1} f(x_N - q_{N-1})(u_1-u_2)^2 \leq 0, \end{aligned}$$

since $\Delta c_{N+1} < 0$. Thus $R_N(\tilde{x}_N)$ is concave in x_N, q_{N-2} and q_{N-1}.

For the $(N-1)^{st}$ period the following profit-to-go function will be written provided that the system state is $\tilde{x}_{N-1}$,

$$\begin{aligned} R_{N-1}(\tilde{x}_{N-1}) &= \max_{y_{N-1} \geq x_{N-1}} \{-c_{N-1}q_{N-1} + p_{N-1}\max(-x_{N-1}, 0) \\ &\quad + p_{N-1}E_{W_{N-1}}\{\min(x_{N-1} + q_{N-1}, W_{N-1})^+\} \\ &\quad - L(x_{N-1} + q_{N-1}) - \alpha\Delta c_{N-1}\min(x_{N-1}, q_{N-2} + q_{N-3})^+ \\ &\quad + \gamma E_{W_{N-1}}(R_N(\tilde{x}_N))\} \end{aligned}$$

By using Lemmas 3, 5 and 6 and after a few arrangements,

$$\begin{aligned} R_{N-1}(\tilde{x}_{N-1}) &= \max_{y_{N-1} \geq x_{N-1}} \{-c_{N-1}y_{N-1} + p_{N-1}(y_{N-1} - \Phi(y_{N-1})) \\ &\quad - L(y_{N-1}) + \gamma E_{W_{N-1}}(R_N(\tilde{x}_N))\} \\ &\quad + p_{N-1}\max(-x_{N-1}, 0) - \alpha\Delta c_{N-1}\min(x_{N-1}, q_{N-2} + q_{N-3})^+ \\ &\quad + c_{N-1}x_{N-1}. \end{aligned}$$

Expected profit-to-go at the $(N-1)^{st}$ period is the following:

$$\begin{aligned} E_{W_{N-1}}(R_N(\tilde{x}_N)) &= \int_0^\infty R_N(\tilde{x}_N)dF(w_{N-1}) \\ &= \int_0^{y_{N-1}-y_N^*} G_N(y_{N-1} - w_{N-1})dF(w_{N-1}) \\ &\quad + G_N(y_N^*)(1 - F(y_{N-1} - y_N^*)) \\ &\quad + \gamma\alpha\Delta c_{N+1}\int_0^\infty \Phi(y_{N-1} - w_{N-1} - y_{N-1} + x_{N-1})dF(w_{N-1}) \\ &\quad + \gamma(p_{N+1} - c_{N+1})\overline{D}_N \\ &\quad - \alpha\Delta c_N \int_0^{y_{N-1}} \min(y_{N-1} - w_{N-1}, y_{N-1} - x_{N-1} + q_{N-2})dF(w_{N-1}) \\ &\quad + p_N\int_0^\infty \max(w_{N-1} - y_N, 0)dF(w_{N-1}) \\ &\quad + c_N\int_0^\infty (y_{N-1} - w_{N-1})dF(w_{N-1}) \end{aligned}$$

$$
\begin{aligned}
&= \int_0^{y_{N-1}-y_N^*} G_N(y_{N-1}-w_{N-1})dF(w_{N-1}) + G_N(y_N^*) \\
&\quad - G_N(y_N^*)F(y_{N-1}-y_N^*) \\
&\quad + \gamma\alpha\Delta c_{N+1}\int_0^{\infty} \Phi(x_{N-1}-w_{N-1})dF(w_{N-1}) \\
&\quad - \alpha\Delta c_N(\Phi(y_{N-1}) - \Phi(x_{N-1}-q_{N-2})) + p_N(\overline{D}_{N-1} + \Phi(y_{N-1})) \\
&\quad + c_N(y_{N-1} - \overline{D}_{N-1}) + \gamma(p_{N+1} - c_{N+1})\overline{D}_N \\
&= \int_0^{y_{N-1}-y_N^*} G_N(y_{N-1}-w_{N-1})dF(w_{N-1}) \\
&\quad - G_N(y_N^*)F(y_{N-1}-y_N^*) \\
&\quad + (-\alpha\Delta c_N + p_N)\Phi(y_{N-1}) + c_N y_{N-1} \\
&\quad + \gamma\alpha\Delta c_{N+1}\int_0^{\infty} \Phi(x_{N-1}-w_{N-1})dF(w_{N-1}) \\
&\quad + \alpha\Delta c_N\Phi(x_{N-1}-q_{N-2}) + (p_N - c_N)\overline{D}_{N-1} \\
&\quad + \gamma(p_{N+1} - c_{N+1})\overline{D}_N + G_N(y_N^*)
\end{aligned}
$$

Then the profit-to-go function at the $(N-1)^{st}$ period is,

$$
\begin{aligned}
R_{N-1}(\tilde{x}_{N-1}) &= \max_{y_{N-1}\geq x_{N-1}} \{G_{N-1}(y_{N-1})\} \\
&\quad + \gamma^2\alpha\Delta c_{N+1}\int_0^{\infty} \Phi(x_{N-1}-w_{N-1})dF(w_{N-1}) \\
&\quad + \gamma\alpha\Delta c_N\Phi(x_{N-1}-q_{N-2}) + \gamma(p_N - c_N)\overline{D}_{N-1} \\
&\quad + \gamma^2(p_{N+1} - c_{N+1})\overline{D}_N + \gamma G_N(y_N^*) \\
&\quad + p_{N-1}\max(-x_{N-1}, 0) - \alpha\Delta c_{N-1}\min(x_{N-1}, q_{N-2} + q_{N-3})^+ \\
&\quad + c_{N-1}x_{N-1}
\end{aligned}
$$

where,

$$
\begin{aligned}
G_{N-1}(y_{N-1}) &= (-c_{N-1} + \gamma c_N + p_{N-1})y_{N-1} \\
&\quad + (-\gamma\alpha\Delta c_N + \gamma p_N - p_{N-1})\Phi(y_{N-1}) \\
&\quad - L(y_{N-1}) + \gamma\int_0^{y_{N-1}-y_N^*} G_N(y_{N-1}-w_{N-1})dF(w_{N-1}) \\
&\quad - \gamma G_N(y_N^*)F(y_{N-1}-y_N^*).
\end{aligned}
$$

For concavity, check the second order condition.

$$
\frac{\partial G_{N-1}(y_{N-1})}{\partial y_{N-1}} = (-c_{N-1} + \gamma c_N + p_{N-1}) + (-\gamma\alpha\Delta c_N + \gamma p_N - p_{N-1})F(y_{N-1})
$$

$$
\begin{aligned}
& -(h_{N-1}+b_{N-1})F(y_{N-1})+b_{N-1} \\
& +\gamma\int_0^{y_{N-1}-y_N^*} G_N^{'}(y_{N-1}-w_{N-1})f(w_{N-1})dw_{N-1} \\
& +\gamma G_N(y_N^*)f(y_{N-1}-y_N^*)-\gamma G_N(y_N^*)f(y_{N-1}-y_N^*) \\
= \;& (-c_{N-1}+\gamma c_N+p_{N-1}) \\
& +(-\gamma\alpha\Delta c_N+\gamma p_N-p_{N-1}-h_{N-1}-b_{N-1})F(y_{N-1}) \\
& +b_{N-1}+\gamma\int_0^{y_{N-1}-y_N^*} G_N^{'}(y_{N-1}-w_{N-1})f(w_{N-1})dw_{N-1},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2 G_{N-1}(y_{N-1})}{\partial y_{N-1}^2} = \;& (-\gamma\alpha\Delta c_N+\gamma p_N-p_{N-1}-h-b)f(y_{N-1}) \\
& +\gamma\int_0^{y_{N-1}-y_N^*} G_N^{''}(y_{N-1}-w_{N-1})f(w_{N-1})dw_{N-1} \\
& +\gamma G_N^{'}(y_N^*)f(y_{N-1}-y_N^*).
\end{aligned}
$$

Since G_N is concave, $G_N^{''}(y_{N-1}-w_{N-1}) \leq 0$ and $G_N^{'}(y_N^*)=0$ due to optimality, therefore if $(-\gamma\alpha\Delta c_N+\gamma p_N-p_{N-1}-h_{N-1}-b_{N-1}) \leq 0$, $G_{N-1}(y_{N-1})$ is concave. The Hessian of $R_{N-1}(\tilde{x}_{N-1})$:

$$
\frac{\partial^2 R_{N-1}(\tilde{x}_{N-1})}{\partial {x_{N-1}}^2} = \gamma^2\alpha\Delta c_{N+1}\int_0^{\infty} f(x_{N-1}-w_{N-1})dF(w_{N-1})+\gamma\alpha\Delta c_N f(x_{N-1}-q_{N-2}),
$$

$$
\frac{\partial^2 R_{N-1}(\tilde{x}_{N-1})}{\partial x_{N-1}\partial q_{N-2}} = -\gamma\alpha\Delta c_N f(x_{N-1}-q_{N-2}),
$$

$$
\frac{\partial^2 R_{N-1}(\tilde{x}_{N-1})}{\partial x_{N-1}\partial q_{N-3}} = \frac{\partial^2 R_{N-1}(\tilde{x}_{N-1})}{\partial q_{N-3}\partial x_{N-1}} = \frac{\partial^2 R_{N-1}(\tilde{x}_{N-1})}{\partial {q_{N-3}}^2} = \frac{\partial^2 R_{N-1}(\tilde{x}_{N-1})}{\partial q_{N-3}\partial q_{N-2}} = \frac{\partial^2 R_{N-1}(\tilde{x}_{N-1})}{\partial q_{N-2}\partial q_{N-3}} = 0,
$$

$$
\frac{\partial^2 R_{N-1}(\tilde{x}_{N-1})}{\partial q_{N-2}\partial x_{N-1}} = -\gamma\alpha\Delta c_N f(x_{N-1}-q_{N-2}),
$$

$$
\frac{\partial^2 R_{N-1}(\tilde{x}_{N-1})}{\partial {q_{N-2}}^2} = \gamma\alpha\Delta c_N f(x_{N-1}-q_{N-2}),
$$

$z = (u_1 \;\; u_2 \;\; u_3)^t$ and let $K = \gamma^2\alpha\Delta c_{N+1}\int_0^{\infty} f(x_{N-1}-w_{N-1})dF(w_{N-1}) \leq 0$ and $A = \gamma\alpha\Delta c_N f(x_{N-1}-q_{N-2}) \leq 0$.

$$
z^t H_{N-1} z = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix} \begin{bmatrix} K+A & 0 & -A \\ 0 & 0 & 0 \\ -A & 0 & A \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}
$$

$$= \begin{bmatrix} u_1(K+A) - u_3A & 0 & (-u_1+u_3)A \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= (u_1(K+A) - u_3A)u_1 + (-u_1A + u_3A)u_3$$

$$= \quad Ku_1^2 + A(u_1 - u_3)^2 \leq 0$$

Thus $R_{N-1}(\tilde{x}_{N-1})$ is concave in x_{N-1}, q_{N-3} and q_{N-2}.

For the $(N-2)^{nd}$ period, the profit-to-go function is,

$$\begin{aligned} R_{N-2}(\tilde{x}_{N-2}) &= \max_{y_{N-2} \geq x_{N-2}} \{-c_{N-2}y_{N-2} + p_{N-2}(y_{N-2} - \Phi(y_{N-2})) \\ &\quad -L(y_{N-2}) + E_{W_{N-2}}(R_{N-1}(\tilde{x}_{N-1}))\} \\ &\quad + p_{N-2}\max(-x_{N-2}, 0) - \alpha\Delta c_{N-2}\min(x_{N-2}, q_{N-4} + q_{N-3})^+ \\ &\quad + c_{N-2}x_{N-2}. \end{aligned}$$

Expected profit-to-go function in the $(N-2)^{nd}$ period is,

$$\begin{aligned} E_{W_{N-2}}(R_{N-1}) &= \int_0^\infty R_{N-1}(\tilde{x}_{N-1})dF(w_{N-2}) \\ &= \int_0^{y_{N-2}-y^*_{N-1}} G_{N-1}(y_{N-2} - w_{N-2})dF(w_{N-2}) \\ &\quad + G_{N-1}(y^*_{N-1})(1 - F(y_{N-2} - y^*_{N-1})) \\ &\quad + \gamma^2\alpha\Delta c_{N+1}\int_0^\infty\int_0^\infty \Phi(y_{N-2} - w_{N-2} - w_{N-1})dF(w_{N-2})dF(w_{N-1}) \\ &\quad + \gamma\alpha\Delta c_N\int_0^\infty \Phi(y_{N-2} - w_{N-2} - (y_{N-2} - x_{N-2}))dF(w_{N-2}) \\ &\quad + \gamma(p_N - c_N)\overline{D}_{N-1} + \gamma^2(p_{N+1} - c_{N+1})\overline{D}_N + \gamma G_N(y^*_N) \\ &\quad + p_{N-1}(\overline{D}_{N-2} + \Phi(y_{N-2})) \\ &\quad - \alpha\Delta c_{N-1}(\Phi(y_{N-2}) - \Phi(x_{N-2} - q_{N-3})) + c_{N-1}y_{N-1} - c_{N-1}\overline{D}_{N-2} \\ &= \int_0^{y_{N-2}-y^*_{N-1}} G_{N-1}(y_{N-2} - w_{N-2})dF(w_{N-2}) \\ &\quad - G_{N-1}(y^*_{N-1})F(y_{N-2} - y^*_{N-1}) \\ &\quad + \gamma^2\alpha\Delta c_{N+1}\int_0^\infty\int_0^\infty \Phi(y_{N-2} - w_{N-2} - w_{N-1})dF(w_{N-1})dF(w_{N-2}) \\ &\quad + (p_{N-1} - \alpha\Delta c_{N-1})\Phi(y_{N-2}) + c_{N-1}y_{N-2} \\ &\quad + \gamma\alpha\Delta c_N\int_0^\infty \Phi(x_{N-2} - w_{N-2})dF(w_{N-2}) \end{aligned}$$

$$
\begin{aligned}
&+ \alpha \Delta c_{N-1} \Phi(x_{N-2} - q_{N-3}) \\
&+ (p_{N-1} - c_{N-1}) \overline{D}_{N-2} + \gamma (p_N - c_N) \overline{D}_{N-1} \\
&+ \gamma^2 (p_{N+1} - c_{N+1}) \overline{D}_N \\
&+ G_{N-1}(y^*_{N-1}) + \gamma G_N(y^*_N).
\end{aligned}
$$

After a few arrangements,

$$
\begin{aligned}
R_{N-2}(\tilde{x}_{N-2}) \;=\; & \max_{y_{N-2} \geq x_{N-2}} \{G_{N-2}(y_{N-2})\} \\
& + \gamma^2 \alpha \Delta c_N \int_0^\infty \Phi(x_{N-2} - w_{N-2}) dF(w_{N-2}) \\
& + \gamma \alpha \Delta c_{N-1} \Phi(x_{N-2} - q_{N-3}) - \alpha \Delta c_{N-2} \min(x_{N-2}, q_{N-4} + q_{N-3})^+ \\
& + p_{N-2} \max(-x_{N-2}, 0) + c_{N-2} x_{N-2} \\
& + \gamma (p_{N-1} - c_{N-1}) \overline{D}_{N-2} + \gamma^2 (p_N - c_N) \overline{D}_{N-1} \\
& + \gamma^3 (p_{N+1} - c_{N+1}) \overline{D}_N \\
& + \gamma G_{N-1}(y^*_{N-1}) + \gamma^2 G_N(y^*_N)
\end{aligned}
$$

where,

$$
\begin{aligned}
G_{N-2}(y_{N-2}) \;=\; & (-c_{N-2} + \gamma c_{N-1} + p_{N-2}) y_{N-2} \\
& + (-\gamma \alpha \Delta c_{N-1} + \gamma p_{N-1} - p_{N-2}) \Phi(y_{N-2}) \\
& - L(y_{N-2}) + \gamma \int_0^{y_{N-2} - y^*_{N-1}} G_{N-1}(y_{N-2} - w_{N-2}) dF(w_{N-2}) \\
& - \gamma G_{N-1}(y^*_{N-1}) F(y_{N-2} - y^*_{N-1}) \\
& + \gamma^3 \alpha \Delta c_{N+1} \int_0^\infty \int_0^\infty \Phi(y_{N-2} - w_{N-2} - w_{N-1}) dF(w_{N-1}) dF(w_{N-2}).
\end{aligned}
$$

The concavity of the $G_{N-2}(y_{N-2})$ and $R_{N-2}(\tilde{x}_{N-2})$ in y_{N-2} and in x_{N-2}, q_{N-3}, q_{N-4} respectively can be shown similar to the $(N-1)^{st}$ case. Assume (i) and (ii) hold for periods $k+1$, $k+2, ..., N.$, then profit-to-go function for the k^{th} period can be written as,

$$
\begin{aligned}
R_k(\tilde{x}_k) \;=\; & \max_{y_k \geq x_k} \big\{ -c_k q_k + p_k \max(-x_k, 0) + p_k E_{W_k} \{ \min(x_k + q_k, W_k)^+ \} \\
& -L(x_k + q_k) - \alpha \Delta c_k \min(x_k, q_{k-2} + q_{k-1})^+ + \gamma E_{W_k}(R_{k+1}(\tilde{x}_{k+1})) \big\} .
\end{aligned}
$$

And by inductive hypothesis the profit-to-go function for the $(k+1)^{st}$ period is,

$$
R_{k+1}(\tilde{x}_{k+1}) \;=\; \max_{y_{k+1} \geq x_{k+1}} \{G_{k+1}(y_{k+1})\}
$$

$$
\begin{aligned}
&+ \gamma^2 \alpha \Delta c_{k+3} \int_0^\infty \Phi(x_{k+1} - w_{k+1}) dF(w_{k+1}) \\
&+ \gamma \alpha \Delta c_{k+2} \Phi(x_{k+1} - q_k) \\
&+ p_{k+1} \max(-x_{k+1}, 0) - \alpha \Delta c_{k+1} \min(x_{k+1}, q_{k-1} + q_k)^+ \\
&+ c_{k+1} x_{k+1} + \left(\sum_{i=k+1}^{i=N-1} \gamma^{i-k} \left((p_{i+1} - c_{i+1}) \overline{D}_i + G_{i+1}(y^*_{i+1}) \right) \right) \\
&+ \gamma^{N-k} c_{N+1} \overline{D}_N .
\end{aligned}
$$

By the use of Lemmas 3, 5 and 6 and after a few arrangements,

$$
\begin{aligned}
R_k(\tilde{x}_k) &= \max_{y_k \geq x_k} \{ -c_k y_k + p_k (y_k - \Phi(y_k)) - L(y_k) + \gamma E_{W_k}(R_{k+1}(\tilde{x}_{k+1})) \} \\
&\quad + p_k \max(-x_k, 0) - \alpha \Delta c_k \min(x_k, q_{k-2} + q_{k-1})^+ + c_k x_k .
\end{aligned}
$$

Expected profit-to-go function in the k^{th} period is,

$$
\begin{aligned}
E_{W_k}(R_{k+1}) &= \int_0^\infty R_{k+1}(\tilde{x}_{k+1}) dF(w_k) \\
&= \int_0^{y_k - y^*_{k+1}} G_{k+1}(y_k - w_k) dF(w_k) \\
&\quad + G_{k+1}(y^*_{k+1})(1 - F(y_k - y^*_{k+1})) \\
&\quad + \gamma^2 \alpha \Delta c_{k+3} \int_0^\infty \int_0^\infty \Phi(y_k - w_k - w_{k+1}) dF(w_k) dF(w_{k+1}) \\
&\quad + \gamma \alpha \Delta c_{k+2} \int_0^\infty \Phi(y_k - w_k - (y_k - x_k)) dF(w_k) \\
&\quad + \left(\sum_{i=k+1}^{i=N-1} \gamma^{i-k} \left((p_{i+1} - c_{i+1}) \overline{D}_i + G_{i+1}(y^*_{i+1}) \right) \right) \\
&\quad + \gamma^{N-k} c_{N+1} \overline{D}_N + p_{k+1} (\overline{D}_k + \Phi(y_k)) \\
&\quad - \alpha \Delta c_{k+1} (\Phi(y_{N-2}) - \Phi(x_k - q_{k-1})) + c_{k+1} y_{k+1} - c_{k+1} \overline{D}_k \\
&= \int_0^{y_k - y^*_{k+1}} G_{k+1}(y_k - w_k) dF(w_k) \\
&\quad - G_{k+1}(y^*_{k+1}) F(y_k - y^*_{k+1}) \\
&\quad + \gamma^2 \alpha \Delta c_{k+3} \int_0^\infty \int_0^\infty \Phi(y_k - w_k - w_{k+1}) dF(w_k) dF(w_{k+1}) \\
&\quad + (p_{k+1} - \alpha \Delta c_{k+1}) \Phi(y_k) + c_{k+1} y_k \\
&\quad + \gamma \alpha \Delta c_{k+2} \int_0^\infty \Phi(x_k - w_k) dF(w_k)
\end{aligned}
$$

$$
\begin{aligned}
&+ \alpha\Delta c_{k+1}\Phi(x_k - q_{k-1}) + G_{k+1}(y^*_{k+1}) \\
&+ (p_{k+1} - c_{k+1})\overline{D}_k \\
&+ \left(\sum_{i=k+1}^{i=N-1} \gamma^{i-k}\left((p_{i+1} - c_{i+1})\overline{D}_i + G_{i+1}(y^*_{i+1})\right)\right) \\
&+ \gamma^{N-k} c_{N+1}\overline{D}_N.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
R_k(\tilde{x}_k) \;=\; & \max_{y_k \geq x_k}\{G_k(y_k)\} \\
&+ \gamma^2\alpha\Delta c_{k+2}\int_0^\infty \Phi(x_k - w_k)dF(w_k) \\
&+ \gamma\alpha\Delta c_{k+1}\Phi(x_k - q_{k-1}) \\
&+ \left(\sum_{i=k}^{i=N-1} \gamma^{i+1-k}\left((p_{i+1} - c_{i+1})\overline{D}_i + G_{i+1}(y^*_{i+1})\right)\right) \\
&+ \gamma^{N+1-k} c_{N+1}\overline{D}_N \\
&\\
&+ p_k\max(-x_k, 0) - \alpha\Delta c_k \min(x_k, q_{k-2} + q_{k-1})^+ + c_k x_k
\end{aligned}
$$

where

$$
\begin{aligned}
G_k(y_k) \;=\; & (-c_k + \gamma c_{k+1} + p_k)y_k + (-\gamma\alpha\Delta c_{k+1} + \gamma p_{k+1} - p_k)\Phi(y_k) \\
&- L(y_k) + \gamma\int_0^{y_k - y^*_{k+1}} G_{k+1}(y_k - w_k)dF(w_k) \\
&- \gamma G_{k+1}(y^*_{k+1})F(y_k - y^*_{k+1}) \\
&+ \gamma^3\alpha\Delta c_{k+3}\int_0^\infty\int_0^\infty \Phi(y_k - w_k - w_{k+1})dF(w_k)dF(w_{k+1}).
\end{aligned}
$$

For concavity check the second order condition.

$$
\begin{aligned}
\frac{\partial G_k(y_k)}{\partial y_k} \;=\; & (-c_k + \gamma c_{k+1} + p_k) + (-\gamma\alpha\Delta c_{k+1} + \gamma p_{k+1} - p_k)F(y_k) \\
&- (h_k + b_k)F(y_k) + b_k + \gamma\int_0^{y_k - y^*_{k+1}} G'_{k+1}(y_k - w_k)f(w_k)dw_k \\
&+ \gamma G_{k+1}(y^*_{k+1})f(y_k - y^*_{k+1}) - \gamma G_{k+1}(y^*_{k+1})f(y_k - y^*_{k+1}) \\
&+ \gamma^3\alpha\Delta c_{k+3}\int_0^\infty\int_0^\infty F(y_k - w_k - w_{k+1})dF(w_k)dF(w_k) \\
=\; & (-c_k + \gamma c_{k+1} + p_k + b_k) + (-\gamma\alpha\Delta c_{k+1} + \gamma p_{k+1} - p_k - h_k - b_k)F(y_k)
\end{aligned}
$$

$$+\gamma\int_0^{y_k-y^*_{k+1}} G'_{k+1}(y_k-w_k)f(w_k)dw_k$$
$$+\gamma^3\alpha\Delta c_{k+3}\int_0^\infty\int_0^\infty F(y_k-w_k-w_{k+1})dF(w_k)dF(w_{k+1}),$$

$$\begin{aligned}\frac{\partial^2 G_k(y_k)}{\partial y_k^2} &= (-\gamma\alpha\Delta c_{k+1}+\gamma p_{k+1}-p_k-h_k-b_k)f(y_k)\\ &\quad+\gamma\int_0^{y_k-y^*_{k+1}} G''_{k+1}(y_k-w_k)f(w_k)dw_k\\ &\quad+\gamma G'_{k+1}(y^*_{k+1})f(y_k-y^*_{k+1})\\ &\quad+\gamma^3\alpha\Delta c_{k+3}\int_0^\infty\int_0^\infty f(y_k-w_k-w_{k+1})dF(w_k)dF(w_{k+1}).\end{aligned}$$

Since G_{k+1} is concave $G''_{k+1}(y_k-w_k)\le 0$ and $G'_{k+1}(y^*_{k+1})=0$ due to optimality, therefore if $(-\gamma\alpha\Delta c_{k+1}+\gamma p_{k+1}-p_k-h-b)\le 0$, $G_k(y_k)$ is concave in y_k.

The Hessian of $R_k(\tilde{x}_k)$ is,

$$\frac{\partial^2 R_k(\tilde{x}_k)}{\partial {x_k}^2}=\gamma^2\alpha\Delta c_{k+2}\int_0^\infty f(x_k-w_k)dF(w_k)+\gamma\alpha\Delta c_{k+1}f(x_k-q_{k-1}),$$
$$\frac{\partial^2 R_k(\tilde{x}_k)}{\partial x_k\partial q_{k-1}}=-\gamma\alpha\Delta c_{k+1}f(x_k-q_{k-1}),$$
$$\frac{\partial^2 R_k(\tilde{x}_k)}{\partial x_k\partial q_{k-2}}=\frac{\partial^2 R_k(\tilde{x}_k)}{\partial q_{k-2}\partial x_k}=\frac{\partial^2 R_k(\tilde{x}_k)}{\partial {q_{k-2}}^2}=\frac{\partial^2 R_k(\tilde{x}_k)}{\partial q_{k-2}\partial q_{k-1}}=\frac{\partial^2 R_k(\tilde{x}_k)}{\partial q_{k-1}\partial q_{k-2}}=0,$$
$$\frac{\partial^2 R_k(\tilde{x}_k)}{\partial q_{k-1}\partial x_k}=-\gamma\alpha\Delta c_{k+1}f(x_k-q_{k-1}),$$
$$\frac{\partial^2 R_k(\tilde{x}_k)}{\partial {q_{k-1}}^2}=\gamma\alpha\Delta c_{k+1}f(x_k-q_{k-1}),$$

$z=(u_1\ \ u_2\ \ u_3)^t$ and let $K=\gamma^2\alpha\Delta c_{k+2}\int_0^\infty f(x_k-w_k)dF(w_k)\le 0$ and $A=\gamma\alpha\Delta c_{k+1}f(x_k-q_{k-1})\le 0$.

$$\begin{aligned}z^tH_{N-1}z &= \begin{bmatrix} u_1 & u_2 & u_3\end{bmatrix}\begin{bmatrix} K+A & 0 & -A\\ 0 & 0 & 0\\ -A & 0 & A\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\\ &= \begin{bmatrix} u_1(K+A)-u_3A & 0 & (-u_1+u_3)A\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}\\ &= (u_1(K+A)-u_3A)u_1+(-u_1A+u_3A)u_3=Ku_1^2+A(u_1-u_3)^2\le 0.\end{aligned}$$

Thus $R_k(\tilde{x}_k)$ is concave in x_k, q_{k-2} and q_{k-1}. Therefore, (iii) of Theorem 4 is proven. The optimal ordering policy of the retailer is a base stock policy.
□

Proof of Theorem 5 We begin by solving the problem from the last period and proceeding backwards for the case where the supplier offers a single protection age limit to the retailer. Profit-to-go function at the N^{th} period is,

$$\begin{aligned} R_N(\tilde{x}_N) &= \min_{q_N \geq 0}\{-c_N q_N + p_N E_{W_N} \min(x_N + q_N, W_k) - L(x_N + q_N) \\ &\quad - \alpha\Delta c_N \min(x_N, q_{N-1})^+ - \gamma\alpha\Delta c_{N+1} E_{W_N}\{\min(q_N, x_N + q_N - W_N)\}^+. \end{aligned}$$

By using Lemma 3 and Lemma 4 the following can be obtained,

$$\begin{aligned} R_N(\tilde{x}_N) &= \min_{y_N \geq x_N}\{-c_N y_N + p_N(y_N - \Phi(y_N)) - L(y_N) \\ &\quad - \alpha\Delta \min(x_N, q_{N-1})^+ - \gamma\alpha\Delta c_{N+1}\left(\Phi(y_N) - \Phi(x_N)\right) + c_N x_N\}. \end{aligned}$$

The profit-to-go function in terms of $G_N(y_N)$ is,

$$R_N(\tilde{x}_N) = \min_{y_N \geq x_N}\{G_N(y_N)\} - \alpha\Delta \min(x_N, q_{N-1})^+ + \gamma\alpha\Delta c_{N+1}\Phi(x_N) + c_N x_N$$

where,

$$G_N(y_N) = y_N(-c_N + p_N) - \Phi(y_N)(p_N - \gamma\alpha\Delta c_{N+1}) - L(y_N).$$

For concavity, check the second order condition.

$$\begin{aligned} \frac{\partial^2 G_N(y_N)}{\partial y_N^2} &= -f(y_N)(p_N - \gamma\alpha\Delta c_{N+1}) - (h_N + b_N)f(y_N) \\ &= f(y_N)(\gamma\alpha\Delta c_{N+1} - p_N - h_N - b_N) \leq 0. \end{aligned}$$

If $\gamma\alpha\Delta c_{N+1} - p_N - h_N - b_N \leq 0$, $G_N(y_N)$ is concave in y_N. Since all of the components of the concavity condition equation is greater or equal to zero, the condition is assured to be less than or equal to zero. Thus $G_N(y_N)$ is concave and attains the maximum at,

$$y_N^* = F^{-1}\left(\frac{p_N - c_N - b_N}{p_N - \alpha\Delta c_{N+1} + h_N + b_N}\right).$$

The Hessian of $R_N(\tilde{x}_N)$ is,

$$z^t H(R_N(\tilde{x}_N))z = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \gamma\alpha\Delta c_{N+1} f(x_N) & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \gamma\alpha\Delta c_{N+1} f(x_N) u_1^2 \leq 0$$

since $\Delta c_{N+1} \leq 0$. Thus $R_N(\tilde{x}_N)$ is concave in q_{N-1} and x_N. Therefore, in the last period order-up-to level of the retailer is shown to be:

$$y_N^* = F^{-1}\left(\frac{p_N - c_N - b_N}{p_N - \alpha\Delta c_{N+1} + h_N + b_N}\right)$$

2-period protection case can be derived similarly.