COMPLETE INTERSECTION MONOMIAL CURVES AND NON-DECREASING HILBERT FUNCTIONS

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By
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July, 2008

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

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ABSTRACT

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In this thesis, we first study the problem of determining set theoretic complete intersection (s.t.c.i.) projective monomial curves. We are also interested in finding the equations of the hypersurfaces on which the monomial curve lie as set theoretic complete intersection. We find these equations for symmetric Arithmetically Cohen-Macaulay monomial curves.

We describe a method to produce *infinitely many* s.t.c.i. monomial curves in $\mathbb{P}^{n+1}$ starting from one single s.t.c.i. monomial curve in $\mathbb{P}^n$. Our approach has the side novelty of describing explicitly the equations of hypersurfaces on which these new monomial curves lie as s.t.c.i.. On the other hand, semigroup gluing being one of the most popular techniques of recent research, we develop numerical criteria to determine when these new curves can or cannot be obtained via gluing.

Finally, by using the technique of gluing semigroups, we give infinitely many new families of affine monomial curves in arbitrary dimensions with Cohen-Macaulay tangent cones. This gives rise to large families of 1-dimensional local rings with arbitrary embedding dimensions and having non-decreasing Hilbert functions. We also construct infinitely many affine monomial curves in $\mathbb{A}^{n+1}$ whose tangent cone is not Cohen Macaulay and whose Hilbert function is non-decreasing from a single monomial curve in $\mathbb{A}^n$ with the same property.

Keywords: monomial curves, complete intersections, toric varieties, tangent cones, Hilbert functions.

ÖZET

TEK TERİMLİ TAM KESİŞİM EĞRİLERİ VE AZALMAYAN HİLBERT FONKSİYONLARI

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Bu tezde ilk olarak projektif uzaydaki tek terimli eğrilerden geometrik tam kesişim olanları tespit etme problemi çalışılmıştır. Ayrıca bir eğriyi geometrik tam kesişim olarak veren hiperyüzeylerin denklemlerini bulma problemi ile de ilgilenilmiştir. Simetrik tek terimli eğrilerden aritmetik olarak Cohen-Macaulay olanlarının, üzerinde tam kesişim olduğu yüzeylerin denklemleri de bulunmuştur.

Bunun yanı sıra, $\mathbb{P}^n$'deki bir geometrik tam kesişim tek terimli eğrisinden $\mathbb{P}^{n+1}$'de *sonsuz tane* geometrik tam kesişim tek terimli eğri üreten bir yöntem geliştirilmiştir. Bu yaklaşımın avantajı, elde edilen yeni eğrileri veren hiperyüzeylerin denklemlerini bulmasıdır. Üretilen eğrilerin, son zamanların en popüler tekniklerinden biri olan yarıgrup birleştirme metoduyla elde edilip edilemeyeceğini kontrol etmek için de sayısal bir ölçüt verilmiştir.

Son olarak, yarıgrup birleştirme metodu kullanılarak, teğet konisi Cohen-Macaulay olan sonsuz yeni afin tek terimli eğri meydana getirilmiştir. Böylece, Hilbert fonksiyonu azalmayan bir boyutlu yerel halkalar elde edilmiştir. Buna ek olarak, $\mathbb{A}^n$'deki Hilbert fonksiyonu azalmayan tek terimli bir eğriden $\mathbb{A}^{n+1}$'de aynı özelliğe sahip ama teğet konu Cohen-Macaulay olmayan sonsuz tek terimli eğri üretilmiştir.

Anahtar sözcükler: tek terimli eğriler, tam kesişimler, torik varyeteler, teğet konileri, Hilbert fonksiyonları.

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Contents

Chapter 1

Introduction

Let K be an algebraically closed field and $K[\mathbf{x}]$ be the polynomial ring $K[x_1, \ldots, x_n]$. To any algebraic variety V of dimension d in $\mathbb{A}^n$, one can associate a prime ideal $I(V) \subset K[\mathbf{x}]$ to be the set of all polynomials vanishing on V. The arithmetical rank of V, denoted by $\mu(V)$, is the least positive integer r for which $I(V) = rad(f_1, \ldots, f_r)$, for some polynomials $f_1, \ldots, f_r$ or equivalently $V = H_1 \bigcap \cdots \bigcap H_r$, where $H_1, \ldots, H_r$ are the hypersurfaces defined by $f_1 = 0, \cdots, f_r = 0$, respectively. We denote by $\mu(I(V))$ the minimal number r for which $I(V) = (f_1, \ldots, f_r)$, for some polynomials $f_1, \ldots, f_r \in R$. These invariants are known to be bounded below by the codimension of the variety (or height of its ideal). So, one has the following relation:

$$n - d \leq \mu(V) \leq \mu(I(V))$$

Although $\mu(I(V))$ has no upper bound (see e.g. [2, 14]), an upper bound for $\mu(V)$ is provided to be n in [20] via commutative algebraic methods. See [71] for a survey on the problem of determining the minimal number of polynomial equations needed to define an algebraic set, which dates back to Kronecker (1882).

The variety V is called a *complete intersection* if $\mu(I(V)) = n - d$. It is called an *almost complete intersection*, if instead, one has $\mu(I(V)) = n - d + 1$. When the arithmetical rank of V takes its lower bound, that is $\mu(V) = n - d$, the variety

V is called a *set-theoretic complete intersection*, s.t.c.i. for short. It is clear that complete intersections are set-theoretic complete intersection. But the converse statement is false as the projective twisted cubic curve is a s.t.c.i. but not a complete intersection curve (cf.[71, Section 4.3.] for details). The corresponding question for almost complete intersection varieties is answered affirmatively in a series of papers by Eto [22, 23, 24] in the case of affine and projective monomial curves over an algebraically closed field of characteristic zero, leaving the general case widely open.

Complete intersection varieties are very special not only because they are the simplest generalizations of hypersurfaces but also they have very special properties. For instance, complete intersection varieties have Gorenstein coordinate rings which are very special Cohen-Macaulay rings. In addition to this, they have proven themselves to be easy to work with. For example, the canonical sheaf of a complete intersection variety V is given easily by a simple formula $\omega_V = \mathcal{O}_V(\sum d_i - n - 1)$, where d_i's are the degrees of the hypersurfaces that cut out the variety V. The multiplicity of the coordinate ring of V has also a simple formula like $\prod d_i$. Another example of this sort is that free resolutions of complete intersections are computed easily via Koszul complexes. So, Hilbert polynomial and genus of a complete intersection variety is estimated rather easily, see [6]. As a special case, if the smooth curve $C \subset \mathbb{P}^3$ is a complete intersection of the smooth surfaces of degrees a and b, then the genus of C is given by $g(C) = \dfrac{1}{2}ab(a + b - 4) + 1$. Therefore, it is worthwhile to investigate which varieties are set theoretic complete intersections including the class of complete intersection varieties.

Determining set-theoretic complete intersection varieties is a classical and longstanding problem in algebraic geometry. Even more difficult is to give explicitly the equations of the hypersurfaces involved. It is believed that the equations of these hypersurfaces or information about them will shed some light on the problem. This is justified by the arose of this kind of papers. For instance, it is shown in [9] that if the hypersurfaces that cut out a s.t.c.i. toric variety are all binomial then the variety is a complete intersection, see also [74]. Another example is that irreducible s.t.c.i. curves on smooth surfaces in $\mathbb{P}^3$ are in fact complete intersections [60]. We know also that if $C \subset \mathbb{A}^3$ is a smooth curve, then its defining ideal $I(C)$ is generated by minors of a matrix of the form

$$\begin{pmatrix} a & c & d \\ b & d & e \end{pmatrix}$$

and C is a set theoretic complete intersection of the surfaces given by $ce - d^2 = 0$ and $a(ae - bd) + b(bc - ad) = 0$, cf. [70]. There are other papers which provide equations or discuss certain properties of the hypersurfaces whose intersection is the variety V, see [7, 8, 35, 42, 48, 72, 68, 76, 78].

There are varieties which are not set theoretic complete intersection. The Segre variety $S = \mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$ is an example for this situation which is given in [43]. Let $t < r < s$ be positive integers, $char(K) = 0$ and $K[x_{ij}]$ be a polynomial ring in rs variables. Then for any t, we have an ideal I_t which defines a non-s.t.c.i. variety, where I_t is the ideal generated by the $t \times t$ minors of the $r \times s$ matrix (x_{ij}), see introduction of [81].

The state of art can be summarized in the most general case as follows. We know that any curve in $\mathbb{A}^n$ is a s.t.c.i. over a field of positive characteristic [16]. In the characteristic zero case, we know only that smooth (more generally locally complete intersection) curves in $\mathbb{A}^n$ are s.t.c.i., see [27, 44]. The same is true for varieties in $\mathbb{A}^n$ if their normal bundles are trivial [10]. It is still an open problem to show that locally complete intersection varieties in $\mathbb{A}^n$ are s.t.c.i. In the projective case, it is known that varieties of dimension at least one which are not connected are not s.t.c.i. [34]. Therefore, the problem is open even for curves in $\mathbb{A}^3$ and for connected curves in $\mathbb{P}^3$.

To study this problem one inevitably tends to choose a special class of (so called toric) varieties. In this case, it is known that all simplicial toric varieties with full parameterization are s.t.c.i. over a field of positive characteristic [8, 35, 48]. On the other hand, nobody knows whether or not the same question has an affirmative answer in the characteristic zero case. However, there are many partial results in this case [11, 12, 25, 36, 39, 52, 58, 62, 63, 77, 78, 79]. In fact, even the case of *symmetric* monomial curves in $\mathbb{P}^3$ is still mysterious.

We are also interested in determining basic properties of the Hilbert function of local rings associated with affine monomial curves. This is worth studying because it gives information about the singularity of the curve. Not much is known about Hilbert functions in the local case. We do not know even when it is non-decreasing. This basic question is studied by several mathematician and Sally states a conjecture saying that one dimensional Cohen-Macaulay rings with small enough embedding dimension have non-decreasing Hilbert functions, [66]. The conjecture is straightforward in the embedding dimension one case, since in this case the local ring is regular and its Hilbert function takes the same value, one, for each variable. The case of embedding dimension two is not trivial and settled by Matlis in [45]. Finally, the case of embedding dimension three, has been proved by Elias in [21]. There are counterexamples to the conjecture in the case of embedding dimension greater than three. The first examples of local rings whose Hilbert function is not non-decreasing were given by Herzog-Waldi [37] and Eakin-Sathaye [19]. These rings are the local rings of affine monomial curves in ten and twelve dimensional spaces respectively. Later, existence of one-dimensional local rings of any embedding dimension greater than four whose Hilbert function is not non-decreasing is proved by Orecchia in [57]. The work [29] of Gupta and Roberts revealed that there are also counterexamples in the case of embedding dimension four. These counterexamples show that the Cohen-Macaulayness of a one-dimensional local ring with embedding dimension greater than three does not guarantee that its Hilbert function is non-decreasing. However, it is a conjecture due to M. E. Rossi, that a one-dimensional Gorenstein local ring (a Cohen-Macaulay ring of type 1) has a non-decreasing Hilbert function. Arslan and Mete has recently proved this conjecture in [4] for Gorenstein local rings with embedding dimension four associated to Gorenstein monomial curves in affine 4-space under a suitable condition. Together with Arslan and Mete, we are interested here in both conjectures in the case of local rings associated to affine monomial curves in any dimensional space.

The organization of the thesis is as follows.

In chapter 2, we introduce a very special family of varieties, so-called toric varieties, which includes affine and projective monomial curves. We discuss some properties of the concepts of projection of toric ideals, gluing toric varieties and extensions of monomial curves, which will be used in the following chapters.

In chapter 3, we pay attention to the symmetric monomial curves in $\mathbb{P}^3$ and classify all arithmetically Cohen-Macaulay monomial curves among them. And then, we give an elementary proof of the fact that they are set theoretic complete intersection by providing explicitly the equations of the surfaces that cut out the curve.

In chapter 4, we develop a method for producing set theoretic complete intersection monomial curves in any dimensional projective space. The method starts with a single s.t.c.i. monomial curve in $\mathbb{P}^n$ and it produces *infinitely* many new s.t.c.i. monomial curves in $\mathbb{P}^{n+1}$. It gives the equations of the hypersurfaces on which new curves lie as s.t.c.i. based on the information provided by the hypersurfaces that defines the curve at the beginning.

In chapter 5, we study the Hilbert function of local rings associated to affine monomial curves. Namely, we use the technique of gluing semigroups to obtain new monomial curves in any dimensional affine space whose Hilbert functions are non-decreasing.

In chapter 6, we discuss some possible continuations of the research carried out in the thesis.

Chapter 2

Toric Varieties and Monomial Curves

Toric varieties arise from different areas of mathematics. They provide a link between Algebraic Geometry, Commutative Algebra, Algebraic Statistics, Number Theory, Graph Theory and Combinatorics. They are important for both theoretical and practical reasons. This is simply because they serve as examples to check validity of many conjectures about more general algebraic varieties. Moreover, the theory of toric varieties provides nice applications to a broad area of mathematics. Certain properties of toric ideals which arise from Graph Theory and Root systems are studied by Ohsugi and Hibi in [53, 54, 55, 56]. Toric varieties coming from Singularity Theory are the subject of the work of Altınok and Tosun in [1] and [80]. Toric varieties arising from Algebraic Statistics are studied by Diaconis and Sturmfels in [18]. For the interaction between Combinatorics and toric varieties, see also [47].

Being a nice and important object, we define and study basic properties of toric varieties in this chapter which will be used later on.

2.1 Toric Variety vs. Toric Set

Let $A = (a_{ij})$ be a $d \times n$ matrix with integer entries whose columns are non-zero. Denote by $\mathbf{a_i} = (a_{1i}, \dots, a_{di})$ the transpose of the i-th column of A and let $\mathcal{A} = \{\mathbf{a_1}, \dots, \mathbf{a_n}\} \subset \mathbb{Z}^d$ be the set of these vectors.

For the sake of simplicity let us denote the polynomial ring $K[x_1, \dots, x_n]$ by $K[\mathbf{x}]$ and the power series ring $K[t_1, \dots, t_d, t_1^{-1}, \dots, t_d^{-1}]$ by $K[\mathbf{t}, \mathbf{t}^{-1}]$. Then, the *toric ideal* I_A (or $I_{\mathcal{A}}$) associated to the matrix A (or the set $\mathcal{A}$, respectively) is defined to be the kernel of the following K-algebra epimorphism:

$$\phi : K[\mathbf{x}] \to K[\mathbf{t}, \mathbf{t}^{-1}], \quad \phi(x_i) := \mathbf{t}^{\mathbf{a_i}}, \quad \text{for all} \quad i = 1, \dots, n.$$

The toric ideal I_A is prime, and thus define an irreducible algebraic set V_A in $\mathbb{A}^n$, called the *affine toric variety* corresponding to A. The dimension of this variety equals the rank of the matrix A.

There are three important algebraic and combinatorial structures related to the toric variety V_A, namely the semigroup $\mathbb{N}\mathcal{A}$, the group $\mathbb{Z}\mathcal{A}$ and the rational polyhedral cone σ_A. We recall that these objects are defined as the sets of vectors which are $\mathbb{N}$-linear, $\mathbb{Z}$-linear and $\mathbb{Q}_{\geq 0}$-linear combinations of elements of $\mathcal{A}$, i.e.

$$\mathbb{N}\mathcal{A} = \{p_1\mathbf{a_1} + \cdots + p_n\mathbf{a_n} | \quad \text{where} \quad p_i \in \mathbb{N}\},$$

$$\mathbb{Z}\mathcal{A} = \{z_1\mathbf{a_1} + \cdots + z_n\mathbf{a_n} | \quad \text{where} \quad z_i \in \mathbb{Z}\} \quad \text{and}$$

$$\sigma_{\mathcal{A}} := pos_{\mathbb{Q}}(\mathcal{A}) = \{q_1\mathbf{a_1} + \cdots + q_n\mathbf{a_n} | \quad \text{where} \quad q_i \in \mathbb{Q}_{\geq 0}\}.$$

The polynomial ring $K[\mathbf{x}]$ is multigraded, i.e. it has more than one grading. One of them is the most natural one where $deg(x_i) = 1$, for all $i = 1, \dots, n$. If $I_{\mathcal{A}}$ is homogeneous with respect to this grading, the variety $V_{\mathcal{A}}$ that it defines lies in $\mathbb{P}^{n-1}$, hence the name *projective toric variety*. The other natural grading is defined as $deg_{\mathcal{A}}(x_i) = \mathbf{a_i} \in \mathcal{A}$. In this case $\mathcal{A}$-degree of a monomial $\mathbf{x}^{\mathbf{u}} := x_1^{u_1} \dots x_n^{u_n}$ becomes a vector:

$$deg_{\mathcal{A}}\, \mathbf{x}^{\mathbf{u}} := u_1\mathbf{a_1} + \cdots + u_n\mathbf{a_n} \in \mathbb{N}\mathcal{A}.$$

The toric ideal $I_{\mathcal{A}}$ is $\mathcal{A}$-homogeneous, that is, all monomials of a polynomial in $I_{\mathcal{A}}$ have the same $\mathcal{A}$-degree. There are also other types of gradings on the polynomial ring $K[\mathbf{x}]$. Indeed, any set $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\} \subset \mathbb{Z}^d$ can be used to grade $K[\mathbf{x}]$ in such a way that $deg_{\mathcal{B}}(x_i) = \mathbf{b_i}$, for $i = 1, \ldots, n$.

There is a strong relation between the elements of the group (or the lattice) $\mathbb{Z}\mathcal{A}$ and the generators of the toric ideal $I_{\mathcal{A}}$. More precisely, $I_{\mathcal{A}}$ is generated by binomials $\mathbf{x^u} - \mathbf{x^v}$, where $\mathbf{u} - \mathbf{v} \in \mathbb{Z}\mathcal{A}$. In terms of Linear Algebra, it can be said that I_A is generated by binomials $\mathbf{x^u} - \mathbf{x^v}$, where $\mathbf{u} - \mathbf{v}$ is an integer vector in the null space of A. Hence, integer matrices whose null spaces contain the same integer vectors give rise to the same toric variety. For a more detailed discussion on generators and Gröbner bases of toric ideals, we refer the reader to [69].

Associated to the matrix A is the *toric set*

$$\Gamma(A) := \{(\mathbf{t^{a_1}}, \ldots, \mathbf{t^{a_n}}) = (t_1^{a_{11}} \cdots t_d^{a_{d1}}, \ldots, t_1^{a_{1n}} \cdots t_d^{a_{dn}}) \quad | \quad t_1, \ldots, t_d \in K\}.$$

We first note that $\Gamma(A) \subset V_A$, since $f(\mathbf{t^{a_1}}, \ldots, \mathbf{t^{a_n}}) = 0$, for any $f \in I_A = Ker(\phi)$. But, in general, the toric set does not parameterize the toric variety, i.e. $\Gamma(A) \neq V_A$. For instance, take

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \end{pmatrix}.$$

Then, it is clear that $I_A = (x_2^2 - x_1x_3)$, since V_A is a toric (hyper)surface in $\mathbb{A}^3$. Obviously, $\Gamma(A) = (t_1t_2^2, t_1^2t_2^3, t_1^3t_2^4)$ and $\Gamma(B) = (s_1, s_1^2s_2, s_1^3s_2^2)$, for $t_1, t_2, s_1, s_2 \in K$. We claim that $\Gamma(A) \neq \Gamma(B) \neq V_A \neq \Gamma(A)$. Observe first that $(0, 0, z) \in V_A$ but it is not an element of the toric sets $\Gamma(A)$ and $\Gamma(B)$, if $z \neq 0$. Similarly $(x, 0, 0)$ is an element of $\Gamma(B)$ but not an element of $\Gamma(A)$, if $x \neq 0$. Hence, a natural question is to determine the conditions under which $V_A = \Gamma(A)$. This is first studied by E. Reyes, R. Villarreal and L. Zarate in [59]. Related to this question is to find a suitable matrix B such that $V_A = \Gamma(B)$. Existence of such a matrix is shown by A. Katsabekis and A. Thoma in [40, 41]. An algorithm is also provided to find a suitable B.

We say that the set $\mathcal{A}$ is a *configuration* if the elements $\mathbf{a_i}$ of $\mathcal{A}$ lie on a hyperplane in $\mathbb{R}^d$. Configurations correspond to projective toric varieties. For instance, consider the set $\mathcal{A} = \{(0, a), (1, b), (2, c)\}$. This set is a configuration if and only if the points $(0, a), (1, b), (2, c)$ are collinear, i.e. they lie on the same line in $\mathbb{R}^2$. Hence, $\mathcal{A}$ is a configuration if and only if $a = 2b - c$. For any integers b and c, we have different configurations $\mathcal{A}_{b,c} = \{(0, 2b - c), (1, b), (2, c)\}$ but we have a unique toric ideal $I_{\mathcal{A}} = (x_2^2 - x_1 x_3)$. Parameterization of the toric variety $V_{\mathcal{A}}$ is given by the configuration $\mathcal{A}_{1,0}$.

There is a special class of toric varieties which are defined and parameterized by the same matrix A, i.e. $V_A = \Gamma(A)$. The form of this matrix is as follows:

$$A = \begin{pmatrix} a_{11} & \cdots & 0 & a_{1(d+1)} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{dd} & a_{d(d+1)} & \cdots & a_{dn} \end{pmatrix}$$

and the parameterization of V_A is $(t_1^{a_{11}}, \ldots, t_d^{a_{dd}}, t_1^{a_{1(d+1)}} \cdots t_d^{a_{d(d+1)}}, \ldots, t_1^{a_{1n}} \cdots t_d^{a_{dn}})$, where $a_{11}, \ldots, a_{dd}$ are positive and the others are non-negative integers, see [40, Corollary 2].

2.2 Monomial Curve

We start with the definition of affine monomial curves. Classically, an *affine monomial curve* in the affine n-space $\mathbb{A}^n$, denoted by $C(m_1, \ldots, m_n)$, is defined parametrically by $(t^{m_1}, \ldots, t^{m_n})$, for some positive integers $m_1 < \cdots < m_n$ with $gcd(m_1, \ldots, m_n) = 1$. This means that if A is a row matrix defined by $A = \begin{pmatrix} m_1 \cdots m_n \end{pmatrix}$ then $I_A = I(C(m_1, \ldots, m_n))$. Monomial curves are *simplicial toric curves* which are parameterized by their toric sets, see [59, Proposition 2.9.]. The condition $gcd(m_1, \ldots, m_n) = 1$ is to ensure that different parameterizations give rise to different toric curves. At the first sight one might think that the parameterization $(t^{gm_1}, \ldots, t^{gm_n})$ defines a simplicial toric curve for each g. But it defines a unique monomial curve $C(m_1, \ldots, m_n)$. To clarify this ambiguity we always assume that $gcd(m_1, \ldots, m_n) = 1$ whenever we talk about monomial curves. The other assumption $m_1 < \cdots < m_n$ in the definition is needed to determine the embedding dimension of the monomial curve, i.e. the dimension of the smallest affine space in which the monomial curve lives. In fact, order of the numbers m_i is not important, the crucial thing here is that they must be different from each other. For instance, embedding dimension of $C = C(1, 2, 2)$ is two, since C is a curve in the plane $x_2 = x_3$ inside $\mathbb{A}^3$. So, the smallest affine space containing C is $\mathbb{A}^2$. Besides, there is no difference between the curves $C(1, 2)$ and $C(2, 1)$, since their geometric properties are the same. Therefore, these assumptions do not harm the generality.

Under the same assumptions on $m_1, \ldots, m_n$, a *projective monomial curve* in $\mathbb{P}^n$, denoted by $\overline{C}(m_1, \ldots, m_n)$, is defined parametrically by

$$(s^{m_n}, s^{m_n-m_1}t^{m_1}, \ldots, s^{m_n-m_{n-1}}t^{m_{n-1}}, t^{m_n}).$$

Note that $\overline{C}(m_1, \ldots, m_n)$ is the projective closures of the affine curves $C(m_1, \ldots, m_n)$ and $C(m_n-m_{n-1}, \ldots, m_n-m_1, m_n)$. Projective monomial curves can be regarded as *simplicial affine toric surfaces* which are parameterized by their toric sets, see [59, Proposition 2.7.].

2.3 Projection of Toric Ideals

First of all, we introduce the geometric notion of projection of rational polyhedral cones and then define the algebraic notion of projection of toric ideals. Let A and B be two integer matrices of size $c \times n$ and $d \times n$. Assume that $dim\ \sigma_A \leq dim\ \sigma_B$ for the corresponding rational convex polyhedral cones σ_A and σ_B. If $\mathcal{A} = \{\mathbf{a_1}, \ldots, \mathbf{a_n}\}$ and $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ are the sets of the column vectors of A and B, then one can define a *projection* $\pi : \sigma_B \to \sigma_A$ of cones via $\pi(\mathbf{b_i}) = \mathbf{a_i}$, for $i = 1, \ldots, n$. For instance, take $\mathcal{A} = \{3, 5, 8\}$ and $\mathcal{B} = \{(1, 2), (2, 1), (3, 3)\}$. Then the map $\pi(y_1, y_2) = (7y_1 + y_2)/3$ defines a projection of the two dimensional polyhedral cone σ_B onto the one dimensional polyhedral cone σ_A. It is not difficult to see that $I_{\mathcal{B}} = (x_1x_2 - x_3)$, $I_{\mathcal{A}} = (x_1x_2 - x_3, x_1^5 - x_2^3)$ and $I_{\mathcal{B}} \subset I_{\mathcal{A}}$. This is not surprising as the following theorem reveals:

Theorem 2.1 *[39, Theorem 2.2] With the preceding notation, the following are equivalent:*

- $I_{\mathcal{B}} \subset I_{\mathcal{A}}$
- *every $\mathcal{B}$-homogeneous ideal in $K[\mathbf{x}]$ is also $\mathcal{A}$-homogeneous*
- *there is a projection of cones $\pi : \sigma_B \to \sigma_A$ given by $\pi(\mathbf{b_i}) = \mathbf{a_i}$, for all $i = 1, \ldots, n$*
- *there is a $c \times d$ matrix D with rational entries such that $DB = A$*

Inspired by the projection of the corresponding cones, Katsabekis in [39] introduced the algebraic notion of *projection*. So, we say that $I_{\mathcal{A}}$ is a projection of $I_{\mathcal{B}}$ if $I_{\mathcal{B}} \subset I_{\mathcal{A}}$. One can study certain algebraic and geometric properties of the toric variety $V_{\mathcal{A}}$ realizing it as a projection of another toric variety $V_{\mathcal{B}}$. A nice example for this situation has been provided in the same paper [39]. For instance, he used the projection of cones $\pi : \sigma_B \to \sigma_A$ and the fact that $V_{\mathcal{B}}$ is a set-theoretic complete intersection to show that $V_{\mathcal{A}}$ is also a set-theoretic complete intersection, where $\mathcal{A} = \{a, a+2b, 2a+3b, 2a+5b\}$ and $\mathcal{B} = \{(5,0),(1,2),(4,3),(0,5)\}$. Katsabekis has studied projections of toric ideals set theoretically. Namely he studied the question of finding suitable polynomials $f_1, \ldots, f_r \in I_{\mathcal{A}}$ such that $rad(I_{\mathcal{A}}) = rad(I_{\mathcal{B}} + (f_1, \ldots, f_r))$. Hence the problem is open ideal theoretically. More precisely, we do not know whether or not we have polynomials $f_1, \ldots, f_r \in I_{\mathcal{A}}$ such that $I_{\mathcal{A}} = I_{\mathcal{B}} + (f_1, \ldots, f_r)$, where $r = \mu(I_A) - \mu(I_B)$.

2.4 Gluing Toric Varieties

Now, we introduce the concept of gluing semigroups. This concept has been introduced for the first time by J. C. Rosales in [65] and used by several authors to produce new examples of set-theoretic and ideal-theoretic complete intersection affine or projective varieties (for example [52], [79]).

Let $\mathcal{A}$ be a subset of $\mathbb{Z}^d$ such that $\mathcal{A} = \mathcal{A}_1 \bigsqcup \mathcal{A}_2$, for some subsets $\mathcal{A}_1$ and $\mathcal{A}_2$. We say that $\mathbb{N}\mathcal{A}$ is a *gluing* of $\mathbb{N}\mathcal{A}_1$ and $\mathbb{N}\mathcal{A}_2$ if there exists a nonzero element $\alpha \in \mathbb{N}\mathcal{A}_1 \bigcap \mathbb{N}\mathcal{A}_2$ such that $\mathbb{Z}\mathcal{A}_1 \bigcap \mathbb{Z}\mathcal{A}_2 = \mathbb{Z}\alpha$. Sometimes we say that the set $\mathcal{A}$ is a gluing of its subsets $\mathcal{A}_1$ and $\mathcal{A}_2$ in the same situation. The crucial benefit of this definition is that we have the following relation between the corresponding toric ideals:

$$I_{\mathcal{A}} = I_{\mathcal{A}_1} + I_{\mathcal{A}_2} + (G_\alpha)$$

where $G_\alpha = M_1 - M_2$ is the relation polynomial and M_i involves variables corresponding to $\mathcal{A}_i$, for details see [79].

Example 2.2 *Let A be the following matrix*

$$\begin{pmatrix} (p+1)m_3 & 0 & 0 & (p+1)(m_3-m_1) & (p+1)(m_3-m_2) & 0 \\ 0 & (p+1)m_3 & 0 & m_1 & m_2 & m_3 \\ 0 & 0 & (p+1)m_3 & pm_1 & pm_2 & pm_3 \end{pmatrix}$$

and $\mathcal{A}$ be the set of its column vectors, where $0 < m_1 < m_2 < m_3$ are integers with $gcd(m_1, m_2, m_3) = 1$ and p is any integer.

Set $\mathcal{A}_1 = \{(0, (p+1)m_3, 0), (0, 0, (p+1)m_3)\}$ and $\mathcal{A}_2 = \mathcal{A} - \mathcal{A}_1$. Then the matrices A_1 and A_2 corresponding to $\mathcal{A}_1$ and $\mathcal{A}_2$ are as follows:

$$A_1 = \begin{pmatrix} 0 & 0 \\ (p+1)m_3 & 0 \\ 0 & (p+1)m_3 \end{pmatrix} \quad \textit{and}$$

$$A_2 = \begin{pmatrix} (p+1)m_3 & (p+1)(m_3-m_1) & (p+1)(m_3-m_2) & 0 \\ 0 & m_1 & m_2 & m_3 \\ 0 & pm_1 & pm_2 & pm_3 \end{pmatrix}.$$

Note that the null space of A_1 is trivial, so $I_{\mathcal{A}_1} = 0$. On the other hand null space of A_2 is the same with the null space of the following matrix

$$B = \begin{pmatrix} m_3 & (m_3-m_1) & (m_3-m_2) & 0 \\ 0 & m_1 & m_2 & m_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad V_B = \overline{C}(m_1, m_2, m_3) \subset \mathbb{P}^3.$$

We observe that $\mathbb{Z}\mathcal{A}_1 \bigcap \mathbb{Z}\mathcal{A}_2 = \mathbb{Z}\alpha$ and the vector α is in $\mathbb{N}\mathcal{A}_1 \bigcap \mathbb{N}\mathcal{A}_2$, where $\alpha = (0, (p+1)m_3, p(p+1)m_3)$. Hence $\mathbb{N}\mathcal{A}$ is a gluing of $\mathbb{N}\mathcal{A}_1$ and $\mathbb{N}\mathcal{A}_2$. If x_i is the variable corresponding to the i-th column vector of A then we have

$$I_{\mathcal{A}} = I_{\mathcal{A}_1} + I_{\mathcal{A}_2} + (x_2 x_3^p - x_6^{p+1}) = I_B + (x_2 x_3^p - x_6^{p+1}).$$

Thus, if $\overline{C}(m_1, m_2, m_3) \subset \mathbb{P}^3$ is a s.t.c.i. on the surfaces X and Y, it readily follows that the toric surface $V_{\mathcal{A}} \subset \mathbb{P}^5$ is a s.t.c.i. on the hypersurfaces X, Y and $x_2 x_3^p = x_6^{p+1}$, for any integer p.

2.5 Extensions of Monomial Curves

Finally, we introduce the concept of extension of monomial curves. This concept is introduced for the first time by Arslan and Mete in [4] in the case of affine monomial curves. Later in [73] we adopt it to the projective case. Thus this section reflects the second and the third sections of [73].

Let m be a positive integer in the numerical semigroup generated by $m_1, \ldots, m_n$, i.e. $m = s_1 m_1 + \cdots + s_n m_n$ where $s_1, \ldots, s_n$ are some non-negative integers. Note that in general there is no unique choice for $s_1, \ldots, s_n$ to represent m in terms of $m_1, \ldots, m_n$. We define the degree $\delta(m)$ of m to be the minimum of all possible sums $s_1 + \cdots + s_n$. If ℓ is a positive integer with $gcd(\ell, m) = 1$, then we say that the monomial curve $\overline{C}(\ell m_1, \ldots, \ell m_n, m)$ in $\mathbb{P}^{n+1}$ is an *extension* of $\overline{C} = \overline{C}(m_1, \ldots, m_n)$. We similarly define $C(\ell m_1, \ldots, \ell m_n, m)$ to be an *extension* of C. We say that an extension is *nice* if $\delta(m) > \ell$ and *bad* otherwise, adopting the terminology of [4].

When the integers $m_1, \ldots, m_n$ are fixed and understood in a discussion, we will use $\overline{C}_{\ell,m}$ to denote the extensions $\overline{C}(\ell m_1, \ldots, \ell m_n, m)$ in $\mathbb{P}^{n+1}$, and use $C_{\ell,m}$ to denote the extensions $C(\ell m_1, \ldots, \ell m_n, m)$ in $\mathbb{A}^{n+1}$.

Extension in the affine case is a special case of gluing. More precisely, if $C_{\ell,m}$ is an extension of C, then the numerical semigroup $< \ell m_1, \ldots, \ell m_n, m >$ is a gluing of $< \ell m_1, \ldots, \ell m_n >$ and $< m >$, as $\mathbb{Z}\{\ell m_1, \ldots, \ell m_n\} \bigcap \mathbb{Z}\{m\} = \mathbb{Z}\{\ell m\}$ with $\ell m \in < \ell m_1, \ldots, \ell m_n > \bigcap < m >$. Thus, we have

$$I(C_{\ell,m}) = I(C) + (x_1^{s_1} \cdots x_n^{s_n} - x_{n+1}^{\ell}).$$

A quick consequence of this is that $C_{\ell,m} \subset \mathbb{A}^{n+1}$ is a s.t.c.i. when $C \subset \mathbb{A}^n$ has the same property.

In the projective case, extension is not always a special case of gluing. There are many projective monomial curves whose underlying affine semigroups can not be obtained by gluing its subsemigroups. This will be studied in details in the section 2.5.2. Now we give a more geometric proof of the fact that extensions of affine s.t.c.i. monomial curves are s.t.c.i. too.

2.5.1 Extensions of Monomial Curves in $\mathbb{A}^n$

Let $C = C(m_1, \ldots, m_n)$ be a s.t.c.i. monomial curve in $\mathbb{A}^n$. In this section, we show that all extensions of C are s.t.c.i. For this we first define, for any ideal $I \subset K[x_1, \ldots, x_{n+1}]$, $\Gamma_\ell(I)$ to be the ideal which is generated by all polynomials of the form $\Gamma_\ell(g)$, where $\Gamma_\ell(g(x_1, \ldots, x_{n+1})) = g(x_1, \ldots, x_n, x_{n+1}^\ell)$, for all $g \in I$. We use the following trick of M. Morales:

Lemma 2.3 ([51, Lemma 3.2]) *Let Y_ℓ be the monomial curve denoted by $C(\ell m_1, \ldots, \ell m_n, m_{n+1})$ in $\mathbb{A}^{n+1}$. Then $I(Y_\ell) = \Gamma_\ell(I(Y_1))$.*

For any extension of C of the form $C_{\ell,m}$, we obviously have $I(C) \subset I(C_{\ell,m})$ and $I(C_{\ell,m}) \cap K[x_1, \ldots, x_n] = I(C)$. The exact relation between the ideals of C and $C_{\ell,m}$ are given by the following lemma.

Lemma 2.4 *Let $m = s_1 m_1 + \cdots + s_n m_n$. For any positive integer ℓ with $gcd(\ell, m) = 1$ we have $I(C_{\ell,m}) = I(C) + (G)$, where $G = {x_1}^{s_1} \cdots\ {x_n}^{s_n} - x_{n+1}^\ell$.*

Proof:

Case $\ell = 1$: We show that $I(C_{1,m}) = I(C) + ({x_1}^{s_1} \cdots\ {x_n}^{s_n} - x_{n+1})$.
For any polynomial $f \in K[x_1, \ldots, x_{n+1}]$, there are polynomials $g \in K[x_1, \ldots, x_n]$ and $h \in K[x_1, \ldots, x_{n+1}]$ such that

$$\begin{aligned} f(x_1, \ldots, x_{n+1}) &= f(x_1, \ldots, x_n, x_{n+1} - x_1^{s_1} \cdots x_n^{s_n} + x_1^{s_1} \cdots x_n^{s_n}) \\ &= g(x_1, \ldots, x_n) + (x_1^{s_1} \cdots x_n^{s_n} - x_{n+1}) h(x_1, \ldots, x_{n+1}). \end{aligned}$$

This identity implies that $f \in I(C_{1,m})$ if and only if $g \in I(C)$.

Case $\ell > 1$**:** Applying Lemma 2.3 with $Y_1 = C_{1,m}$ we have

$$\begin{aligned} I(C_{\ell,m}) &= \Gamma_\ell(I(C_{1,m})), \text{ by Lemma 2.3} \\ &= \Gamma_\ell(I(C) + (x_1^{s_1} \cdots x_n^{s_n} - x_{n+1})) \text{ by the first part of this lemma} \\ &= I(C) + (G). \end{aligned}$$

□

This lemma provides an alternate proof to the following theorem which is a special case of [79, Theorem 2].

Theorem 2.5 *If $C \subset \mathbb{A}^n$ is a s.t.c.i. monomial curve, then all extensions of the form $C_{\ell,m} \subset \mathbb{A}^{n+1}$ are also s.t.c.i. monomial curves.*

Proof: Since $I(C_{\ell,m}) = I(C) + (G)$ by Lemma 2.4, it follows that

$$\begin{aligned} Z(I(C_{\ell,m})) &= Z(I(C) + (G)) \\ C_{\ell,m} &= Z(I(C)) \bigcap Z(G), \end{aligned}$$

where $Z(\cdot)$ denotes the zero set as usual. Hence $C_{\ell,m}$ is a s.t.c.i. if C is. □

2.5.2 Extensions That Can Not Be Obtained By Gluing

If $\overline{C}(m_1, \ldots, m_{n+1})$ is a monomial curve in $\mathbb{P}^{n+1}$, then there is a corresponding semigroup $\mathbb{N}T$, where

$$T = \{(m_{n+1}, 0), (m_{n+1} - m_1, m_1), \ldots, (m_{n+1} - m_n, m_n), (0, m_{n+1})\} \subset \mathbb{N}^2.$$

Let $T = T_1 \bigsqcup T_2$ be a decomposition of T into two disjoint proper subsets. Without loss of generality assume that the cardinality of T_1 is less than or equal to the cardinality of T_2. $\mathbb{N}T$ is called a *gluing* of $\mathbb{N}T_1$ and $\mathbb{N}T_2$ if there exists a nonzero $\alpha \in \mathbb{N}T_1 \bigcap \mathbb{N}T_2$ such that $\mathbb{Z}\alpha = \mathbb{Z}T_1 \bigcap \mathbb{Z}T_2$. Following the literature we write $I(T)$ for the ideal of the toric variety corresponding to the affine semigroup $\mathbb{N}T$. Note that if $\mathbb{N}T$ is a gluing of $\mathbb{N}T_1$ and $\mathbb{N}T_2$ then we have $I(T) = I(T_1) + I(T_2) + (G_\alpha)$, where G_α is the relation polynomial, see [79].

We note that the condition $\mathbb{Z}\alpha = \mathbb{Z}T_1 \bigcap \mathbb{Z}T_2$ is not fulfilled when T_1 is not a singleton. Hence we formulate this observation to be the following

Proposition 2.6 *If T_1 is not a singleton then $\mathbb{N}T$ is not a gluing of $\mathbb{N}T_1$ and $\mathbb{N}T_2$.*

Proof: If T_1 is not a singleton, then neither is T_2 by the assumption on the cardinalities of these sets. Thus $\mathbb{Z}T_1$ and $\mathbb{Z}T_2$ are submodules of $\mathbb{Z}^2$ of rank two each. It is elementary to show that their intersection has rank two. For instance, let r and t be generators of $\mathbb{Z}T_1$, then the images of r and t have finite order in the finite group $\mathbb{Z}^2/\mathbb{Z}T_2$, meaning that ar and bt are in $\mathbb{Z}T_2$ for some positive integers a and b. Then the rank two $\mathbb{Z}$-module generated by ar and bt is contained in the intersection $\mathbb{Z}T_1 \cap \mathbb{Z}T_2$ which must be of rank two itself being a submodule of $\mathbb{Z}^2$.

Hence the intersection cannot be generated by a single element. Thus $\mathbb{N}T$ is not a gluing of $\mathbb{N}T_1$ and $\mathbb{N}T_2$. □

This proposition means that the only way to show that an extension in $\mathbb{P}^{n+1}$ is a s.t.c.i. via gluing is to apply the technique to a projective monomial curve in $\mathbb{P}^n$. Thus we discuss the case where T_1 is a singleton. But if T_1 is $\{(m_{n+1}, 0)\}$ or $\{(0, m_{n+1})\}$ then $\mathbb{N}T_1 \bigcap \mathbb{N}T_2 = \{(0,0)\}$. So it is sufficient to deal with the case where T_1 is of the form $\{(m_{n+1} - m_i, m_i)\}$, for some $i \in \{1, \ldots, n\}$.

From now on, Δ_i denotes the greatest common divisor of the positive integers $m_1, \ldots, \widehat{m_i}, \ldots, m_{n+1}$ (m_i is omitted), for $i = 1, \ldots, n$. Note that we have $gcd(\Delta_i, m_i) = 1$, for all $i = 1, \ldots, n$, since $gcd(m_1, \ldots, m_{n+1}) = 1$.

Proposition 2.7 *If $T_1 = \{(m_{n+1} - m_{i_0}, m_{i_0})\}$ for some fixed $i_0 \in \{1, \ldots, n\}$, then $\mathbb{N}T$ is a gluing of $\mathbb{N}T_1$ and $\mathbb{N}T_2$ if and only if there exist non-negative integers d_j, for $j = 1, \ldots, \widehat{i_0}, \ldots, n+1$, satisfying the following two conditions:*

$$\text{(I) } \Delta_{i_0} m_{i_0} = \sum_{j=1\,(j\neq i_0)}^{n+1} d_j m_j, \quad \text{and} \quad \text{(II) } \Delta_{i_0} \geq \sum_{j=1\,(j\neq i_0)}^{n+1} d_j.$$

Proof: Let $\alpha = \Delta_{i_0}(m_{n+1} - m_{i_0}, m_{i_0})$. We first show that $\mathbb{Z}T_1 \bigcap \mathbb{Z}T_2 = \mathbb{Z}\alpha$. Since $\Delta_{i_0} = gcd(m_1, \ldots, \widehat{m_{i_0}}, \ldots, m_{n+1})$, there are $z_j \in \mathbb{Z}$, for $j = 1, \ldots, \widehat{i_0}, \ldots, n+1$, such that $\Delta_{i_0} = \sum_{j\neq i_0} z_j m_j$. So, $\Delta_{i_0} m_{i_0} = \sum_{j\neq i_0} m_{i_0} z_j m_j$ which implies that

$$\Delta_{i_0}(m_{n+1} - m_{i_0}, m_{i_0}) = \sum_{j\neq i_0} m_{i_0} z_j (m_{n+1} - m_j, m_j) + (\Delta_{i_0} - \sum_{j\neq i_0} m_{i_0} z_j)(m_{n+1}, 0).$$

Thus $\alpha = \Delta_{i_0}(m_{n+1} - m_{i_0}, m_{i_0}) \in \mathbb{Z}T_1 \bigcap \mathbb{Z}T_2$ implying $\mathbb{Z}\alpha \subseteq \mathbb{Z}T_1 \bigcap \mathbb{Z}T_2$.

For the converse inclusion, take $c(m_{n+1} - m_{i_0}, m_{i_0}) \in \mathbb{Z}T_1 \bigcap \mathbb{Z}T_2$, for some $c \in \mathbb{Z}$. Then, obviously we have $c(m_{n+1} - m_{i_0}, m_{i_0}) \in \mathbb{Z}T_2$ which implies that $cm_{i_0} \in \mathbb{Z}(\{m_1, \ldots, \widehat{m_{i_0}}, \ldots, m_{n+1}\}) = \mathbb{Z}\Delta_{i_0}$. So, Δ_{i_0} divides cm_{i_0}. If $\Delta_{i_0} > 1$, then Δ_{i_0} divides c, since it does not divide m_{i_0} (remember that $gcd(\Delta_{i_0}, m_{i_0}) = 1$). If $\Delta_{i_0} = 1$, obviously Δ_{i_0} divides c. Thus, $c(m_{n+1} - m_{i_0}, m_{i_0})$ is a multiple of α and $\mathbb{Z}T_1 \bigcap \mathbb{Z}T_2 \subseteq \mathbb{Z}\alpha$.

Since $\mathbb{Z}T_1 \bigcap \mathbb{Z}T_2 = \mathbb{Z}\alpha$, it will follow by definition that $\mathbb{N}T$ is a gluing of $\mathbb{N}T_1$ and $\mathbb{N}T_2$ if and only if $\alpha \in \mathbb{N}T_1 \bigcap \mathbb{N}T_2$. But, if $\alpha \in \mathbb{N}T_1 \bigcap \mathbb{N}T_2$ then there exists non-negative integers d_j and d for which we have

$$\begin{aligned} \Delta_{i_0}(m_{n+1} - m_{i_0}, m_{i_0}) &= \sum_{j\neq i_0} d_j (m_{n+1} - m_j, m_j) + d(m_{n+1}, 0) \\ (\Delta_{i_0} m_{n+1} - \Delta_{i_0} m_{i_0}, \Delta_{i_0} m_{i_0}) &= ([d + \sum_{j\neq i_0} d_j] m_{n+1} - \sum_{j\neq i_0} d_j m_j, \sum_{j\neq i_0} d_j m_j). \end{aligned}$$

Thus, $\Delta_{i_0} m_{i_0} = \sum_{j\neq i_0} d_j m_j$ and $d = \Delta_{i_0} - \sum_{j\neq i_0} d_j$. Since $d \geq 0$, we see that the conditions (I) and (II) hold. On the other hand, if (I) and (II) hold then we observe that $\alpha \in \mathbb{N}T_1 \bigcap \mathbb{N}T_2$, by the equalities above. Thus, the condition $\alpha \in \mathbb{N}T_1 \bigcap \mathbb{N}T_2$ is equivalent to the existence of the non-negative integers d_j satisfying (I) and (II). $\square$

As a direct consequence of Proposition 2.7 we get the following

Corollary 2.8 *If $\Delta_{i_0} = 1$, for some fixed $i_0 \in \{1, \ldots, n\}$, then $\mathbb{N}T$ cannot be obtained as a gluing of $\mathbb{N}T_1$ and $\mathbb{N}T_2$, where $T_1 = \{(m_{n+1} - m_{i_0}, m_{i_0})\}$ and $T_2 = T - T_1$.*

Proof: We apply Proposition 2.7. If (I) does not hold, we are done. If it holds, then we have two cases: either $\sum_{j=1\,(j\neq i_0)}^{n+1} d_j = 1$ or $\sum_{j=1\,(j\neq i_0)}^{n+1} d_j > 1$. The first case forces $m_{i_0} = m_j$ for some $j \neq i_0$, from (I), but this contradicts the way we choose $m_i's$. The second case causes (II) to fail, as $\Delta_{i_0} = 1$. □

Example 2.9 *If we consider the curve $\overline{C}(2,3,4,8) \subset \mathbb{P}^4$ and take $i_0 = 2$, then the conditions* (I) *and* (II) *of the above proposition hold. Thus this curve can be obtained by gluing.*

But if we consider the monomial curve $\overline{C}(2,4,7,8) \subset \mathbb{P}^4$, then for every choice of i_0, either $\Delta_{i_0} = 1$, or else condition (II) *of the above proposition fails. Hence this curve cannot be obtained by gluing.*

Corollary 2.10 *Let $\overline{C}_{\ell,m} \subset \mathbb{P}^{n+1}$ be a bad extension of $\overline{C} = \overline{C}(m_1, \ldots, m_n)$, i.e. $\ell \geq \delta(m)$. If $\overline{C}$ is a s.t.c.i. on the hypersurfaces $f_1 = \cdots = f_{n-1} = 0$, then $\overline{C}_{\ell,m}$ can be shown to be a s.t.c.i. on the hypersurfaces $f_1 = \cdots = f_{n-1} = 0$ and $F = x_{n+1}^{\ell} - x_0^{\ell-\delta(m)} x_1^{s_1} \cdots x_n^{s_n} = 0$ by the technique of gluing, where $m = s_1 m_1 + \cdots + s_n m_n$ and $s_1 + \cdots + s_n = \delta(m)$.*

Proof: Since $m_1 < \cdots < m_n$ and $m = s_1 m_1 + \cdots + s_n m_n \leq \delta(m) m_n \leq \ell m_n$, it follows that ℓm_n is the biggest number among $\{\ell m_1, \ldots, \ell m_n, m\}$. The extension $\overline{C}_{\ell,m}$ corresponds to the semigroup $\mathbb{N}T$, where $T = T_1 \bigcup T_2$, $T_1 = \{(\ell m_n - m, m)\}$ and $T_2 = \{(\ell m_n, 0), (\ell m_n - \ell m_1, \ell m_1), \ldots, (\ell m_n - \ell m_{n-1}, \ell m_{n-1}), (0, \ell m_n)\}$. Since $gcd(\ell m_1, \ldots, \ell m_n) = \ell$, $\ell m = s_1(\ell m_1) + \cdots + s_n(\ell m_n)$ and $\ell \geq \delta(m)$, $\mathbb{N}T$ is a gluing of $\mathbb{N}T_1$ and $\mathbb{N}T_2$, by Proposition 2.7. Since $I(T) = I(T_1) + I(T_2) + (F)$, the claim follows from [79, Theorem 2]. □

Chapter 3

Symmetric Monomial Curves in $\mathbb{P}^3$

The purpose of this chapter is to give an alternative proof of the fact that symmetric monomial curves in $\mathbb{P}^3$ which are arithmetically Cohen-Macaulay are s.t.c.i. by elementary algebraic methods inspired by [11]. The proof is constructive and provides the equations of the hypersurfaces cutting out the curve.

Let $p < q < r$ be some positive integers. Recall that a monomial curve $\overline{C}(p, q, r)$ in $\mathbb{P}^3$ is given parametrically by

$$(w, x, y, z) = (u^r, u^{r-p}v^p, u^{r-q}v^q, v^r)$$

where $(u, v) \in \mathbb{P}^1$. It can be seen that $\overline{C}(p, q, r)$ is a smooth curve if and only if it is of the form $\overline{C}(1, q, q + 1)$. No smooth curve of this form is known to be s.t.c.i. except the twisted cubic (for which $q = 2$). They can not be s.t.c.i. on smooth surfaces, see [38].

We say that the monomial curve $\overline{C}(p, q, r)$ is *symmetric* if $p + q = r$. In this case the parametric representation of the curve $\overline{C}(p, q, p + q)$ becomes

$$(u^{p+q}, u^q v^p, u^p v^q, v^{p+q}).$$

It is known that all monomial curves are s.t.c.i. in $\mathbb{P}^3$, if the base field K is of positive characteristic, [35]. But, no one knows whether even the symmetric monomial curves are s.t.c.i. in $\mathbb{P}^3$ in the characteristic zero case. To address this case, we work with an algebraically closed field K of characteristic zero, throughout the chapter.

It is not difficult to show that symmetric monomial curves $\overline{C}(p,q,p+q) \subset \mathbb{P}^3$ can not be s.t.c.i. on the smooth quadric $Q : xy = zw$. We will achieve this result by showing that $\overline{C}$ is of type (p,q) on Q and that complete intersections on Q is of type (d,d), for some d.

Claim: $\overline{C} = \overline{C}(p,q,p+q) \subset \mathbb{P}^3$ is of type (p,q) on Q.

Proof: Recall that Q is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, see [33, Ex.I.2.15]. More precisely, it is the image of the following map:

$$\psi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3, \quad \psi((a_0,a_1) \times (b_0,b_1)) = (a_0b_0, a_0b_1, a_1b_0, a_1b_1).$$

We have two families of lines L and M on Q, defined by:

$$\begin{aligned} L_\infty &:= \psi((0,1) \times (b_0,b_1)) = (0,0,b_0,b_1) \\ L_t &:= \psi((1,t) \times (b_0,b_1)) = (b_0,b_1,tb_0,tb_1), \quad \text{where} \quad t \in K. \end{aligned}$$

and

$$\begin{aligned} M_\infty &:= \psi((a_0,a_1) \times (0,1)) = (0,a_0,0,a_1) \\ M_u &:= \psi((a_0,a_1) \times (1,u)) = (a_0,ua_0,a_1,ua_1), \quad \text{where} \quad u \in K. \end{aligned}$$

Picard group of Q is generated by L and M, so type of a curve on Q is determined by the intersection of the curve with L and M. To see that $\overline{C}$ is of type (p,q), we need to observe that $\overline{C} \cdot M_u = p$ and $\overline{C} \cdot L_t = q$.

Note that $(u^{p+q}, u^q v^p, u^p v^q, v^{p+q}) = (b_0, b_1, tb_0, tb_1)$ is a point of the intersection $\overline{C} \bigcap L_t$. Since $(b_0,b_1) \neq (0,0)$, we have $u = 1$ and thus $b_0 = 1$ and $t = v^q$. Thus we have a point $(1, v^p, v^q, v^{p+q}) = (1, b_1, t, tb_1)$ in the intersection with multiplicity q.

Similarly, $(u^{p+q}, u^q v^p, u^p v^q, v^{p+q}) = (a_0, ua_0, a_1, ua_1)$ is a point of the intersection $\overline{C} \bigcap M_u$. Since $(a_0,a_1) \neq (0,0)$, we have $u = 1$ and thus $a_0 = 1$ and $u = v^p$.

Thus we have a point $(1, v^p, v^q, v^{p+q}) = (1, u, a_1, ua_1)$ in the intersection with multiplicity p. $\square$

The following more general result implies that complete intersections on Q is of type (d, d), since H has type $(1, 1)$, where H is the hyperplane defined by $x = 0$.

Proposition 3.1 *C is the complete intersection of the smooth surface X_s of degree s and the surface V_d of degree d if and only if $C \sim dH$, where H is a hyperplane section of X_s.*

Proof: Let us assume that C is a complete intersection of X_s and V_d. Since $V_d \sim d\mathbb{P}^2$ and $H = X_s \bigcap \mathbb{P}^2$, it follows that

$$C = X_s \bigcap V_d \sim X_s \bigcap d\mathbb{P}^2 = dH.$$

On the other hand, if $C \sim dH$ then obviously C is a complete intersection of X_s and V_d. To see this consider the following exact sequence:

$$0 \to Q_{\mathbb{P}^3}(d - s) \to Q_{\mathbb{P}^3}(d) \to Q_X(d) \to 0$$

By taking the cohomology of each term, we get the following long exact sequence:

$$0 \to H^0(Q_{\mathbb{P}^3}(d - s)) \to H^0(Q_{\mathbb{P}^3}(d)) \to H^0(Q_X(d)) \to$$

$$\to H^1(Q_{\mathbb{P}^3}(d - s)) \to H^1(Q_{\mathbb{P}^3}(d)) \to H^1(Q_X(d)) \to \ldots$$

Since $H^i(\mathbb{P}^3, Q_{\mathbb{P}^3}(d)) = 0$ for $0 < i < 3$ and $d \in \mathbb{Z}$, it follows that

$$0 \to H^0(Q_{\mathbb{P}^3}(d - s)) \to H^0(Q_{\mathbb{P}^3}(d)) \to H^0(Q_X(d)) \to 0$$

i.e $H^0(Q_{\mathbb{P}^3}(d)) \to H^0(Q_X(d))$ is surjective.

Thus a section f, defining the curve $C \sim dH$, is the restriction of a section F on X_s. If $V_d = Z(F)$, C is the complete intersection $X_s \bigcap V_d$. $\square$

Corollary 3.2 *$\overline{C}(p, q, p + q) \subset \mathbb{P}^3$ can not be s.t.c.i. on $Q : xy = zw$.*

Proof: Assume that $\overline{C} = \overline{C}(p, q, p+q)$ is a s.t.c.i. of Q and V_d. Then we have $Q \bigcap V_d = k\overline{C}$, for some k. Since type of the complete intersection $Q \bigcap V_d$ is (d, d) and the type of $\overline{C}$ is (p, q), we have $(d, d) = k(p, q)$, which has no solution for k. Contradiction. $\square$

A *minimal* system of generators for the ideal of symmetric monomial curves in $\mathbb{P}^3$ is given in [13] as follows:

$$f = xy - wz \quad \text{and} \quad F_i = w^{q-p-i}y^{p+i} - x^{q-i}z^i, \quad \text{for all} \quad 0 \leq i \leq q-p.$$

Recall that a monomial curve $\overline{C}(p, q, r) \subset \mathbb{P}^3$ is called Arithmetically Cohen-Macaulay (ACM) if its projective coordinate ring is Cohen-Macaulay. In the same article [13], it is also proven that a monomial curve in $\mathbb{P}^3$ is ACM if and only if its ideal is generated by at most 3 polynomials. Now, if the ideal of a symmetric monomial curve $\overline{C}(p, q, p+q)$ is generated by two polynomials it would follow that $p = q$. But, this contradicts with the assumption that $p < q < r$. So, the ideal of an ACM symmetric monomial curve $\overline{C}(p, q, p+q)$ is generated by three polynomials and hence $p = q-1$, where necessarily $q > 1$. Thus, all symmetric ACM monomial curves in $\mathbb{P}^3$ are of the form $\overline{C}(q-1, q, 2q-1)$ and their defining ideals are generated minimally by the following three polynomials:

$$\begin{aligned} f &= xy - zw, \\ g : &= -F_1 = x^{q-1}z - y^q, \\ h : &= -F_0 = x^q - y^{q-1}w. \end{aligned}$$

The fact that $\overline{C}(q-1, q, 2q-1)$ is a s.t.c.i. curve was shown in [63], but the equation of the second surface was not given. Here, we give an alternative proof that constructs the polynomial G such that the symmetric ACM monomial curve is the intersection of the surface $G = 0$ and a binomial surface defined by one of f, g and h. We construct G by adding $x^q g$ to the q-th power of f and dividing the sum by z. Hence we get the following theorem in [72]:

Theorem 3.3 *Any symmetric Arithmetically Cohen-Macaulay monomial curve in $\mathbb{P}^3$, which is given by $\overline{C}(q-1, q, 2q-1)$ for some $q > 1$, is a set-theoretic* *complete intersection of the following two surfaces*

$$g = x^{q-1}z - y^q = 0 \quad \textit{and}$$

$$G = x^{2q-1} + \sum_{k=1}^{q} (-1)^k \frac{q!}{(q-k)!k!} x^{q-k} y^{q-k} z^{k-1} w^k = 0.$$

Proof: Note first that $zG = f^q + x^q g$. Take a point (w_0, x_0, y_0, z_0) from $Z(f, g, h)$. Then, by $z_0 G(w_0, x_0, y_0, z_0) = f^q(w_0, x_0, y_0, z_0) + x_0^q g(w_0, x_0, y_0, z_0) = 0$ we observe that either $G(w_0, x_0, y_0, z_0) = 0$ or $z_0 = 0$.

If $G(w_0, x_0, y_0, z_0) = 0$ then $(w_0, x_0, y_0, z_0) \in Z(g, G)$. If $z_0 = 0$ then by $g(w_0, x_0, y_0, z_0) = 0$ we get $y_0 = 0$, and by $h(w_0, x_0, y_0, z_0) = 0$ we get $x_0 = 0$. Thus $(w_0, x_0, y_0, z_0) = (1, 0, 0, 0)$ which is in $Z(g, G)$.

Let us now take a point $(w_0, x_0, y_0, z_0) \in Z(g, G)$. Then either $z_0 = 0$ or we can assume $z_0 = 1$. If $z_0 = 0$ then by $g(w_0, x_0, y_0, z_0) = 0$ we get $y_0 = 0$, and by $G(w_0, x_0, y_0, z_0) = 0$ we obtain $x_0 = 0$ in this case. Thus we get the point $(w_0, x_0, y_0, z_0) = (1, 0, 0, 0)$ which is in $Z(f, g, h)$. On the other hand, if $z_0 = 1$ then by $G = f^q + x_0^q g$ we see that $f(w_0, x_0, y_0, z_0) = 0$. Moreover, we have $x_0 y_0 = w_0$ and $x_0^{q-1} = y_0^q$ in this case. Hence we obtain the following $x_0^q = x_0 x_0^{q-1} = x_0 y_0^q = x_0 y_0 y_0^{q-1} = w_0 y_0^{q-1}$, meaning that $h(w_0, x_0, y_0, z_0) = 0$. $\square$

Note that the symmetric ACM monomial curves above are s.t.c.i. on the binomial surface $g = 0$. This is not true for symmetric non-ACM monomial curves, that is, they can never be a s.t.c.i. on a binomial surface, [75, Theorem 5.1]. Thus it is very difficult to construct hypersurfaces on which symmetric non-ACM monomial curves in $\mathbb{P}^3$ are s.t.c.i. with the simplest open case being the Macaulay's quartic curve $\overline{C}(1, 3, 4)$.

Chapter 4

Producing S.T.C.I. Monomial Curves in $\mathbb{P}^n$

The aim of this chapter is to study nice extensions of projective monomial curves and follows the fourth and the fifth section of [73]. Since the relation between the ideal of the curve and that of its nice extensions are not known explicitly, we use the information provided by their affine parts here. So we need frequently to refer to the Section 2.5. Let us recall the notation there.

Throughout the chapter, K will be assumed to be an algebraically closed field of characteristic zero. By an *affine monomial curve* $C(m_1, \ldots, m_n)$, for some positive integers $m_1 < \cdots < m_n$ with $gcd(m_1, \ldots, m_n) = 1$, we mean a curve with generic zero $(v^{m_1}, \ldots, v^{m_n})$ in the affine n-space $\mathbb{A}^n$, over K. By a *projective monomial curve* $\overline{C}(m_1, \ldots, m_n)$ we mean a curve with generic zero

$$(u^{m_n}, u^{m_n - m_1} v^{m_1}, \ldots, u^{m_n - m_{n-1}} v^{m_{n-1}}, v^{m_n})$$

in the projective n-space $\mathbb{P}^n$, over K. We use the fact that $\overline{C}(m_1, \ldots, m_n)$ is the projective closure of $C(m_1, \ldots, m_n)$.

Whenever we write $\overline{C} \subset \mathbb{P}^n$ to simplify the notation, we always mean a monomial curve $\overline{C}(m_1, \ldots, m_n)$ for some fixed positive integers $m_1 < \cdots < m_n$ with $gcd(m_1, \ldots, m_n) = 1$.

Let m be a positive integer in the numerical semigroup generated by $m_1, \ldots, m_n$, i.e. $m = s_1 m_1 + \cdots + s_n m_n$ where $s_1, \ldots, s_n$ are some non-negative integers. We define the degree $\delta(m)$ of m to be the minimum of all possible sums $s_1 + \cdots + s_n$. If ℓ is a positive integer with $gcd(\ell, m) = 1$, then we say that the monomial curve $\overline{C}(\ell m_1, \ldots, \ell m_n, m)$ in $\mathbb{P}^{n+1}$ is an *extension* of $\overline{C}$. An extension is *nice* if $\delta(m) > \ell$ and *bad* otherwise.

Recall that $\overline{C}_{\ell,m}$ denotes the extensions $\overline{C}(\ell m_1, \ldots, \ell m_n, m)$ in $\mathbb{P}^{n+1}$, and $C_{\ell,m}$ denotes the extensions $C(\ell m_1, \ldots, \ell m_n, m)$ in $\mathbb{A}^{n+1}$.

4.1 Nice Extensions of Monomial Curves

Since *bad* extensions are shown to be a s.t.c.i. by the technique of gluing (see Corollary 2.10), we study *nice* extensions of monomial curves in this section. By using the theory developed in section 2.5.2 one can check which of these extensions can be obtained by the technique of gluing semigroups.

Throughout this section we will assume that

- $\overline{C} = \overline{C}(m_1, \ldots, m_n) \subset \mathbb{P}^n$ is a s.t.c.i. on $f_1 = \cdots = f_{n-1} = 0$
- $m = s_1 m_1 + \cdots + s_n m_n$ for some nonnegative integers $s_1, \ldots, s_n$ such that $s_1 + \cdots + s_n = \delta(m)$
- ℓ is a positive integer with $gcd(\ell, m) = 1$
- $\delta(m) > \ell$.

Remark 4.1 *Since $\overline{C}$ is s.t.c.i. on $f_1 = \cdots = f_{n-1} = 0$, its affine part C is s.t.c.i. on $g_1 = \cdots = g_{n-1} = 0$, where $g_i(x_1, \ldots, x_n) = f_i(1, x_1, \ldots, x_n)$ is the dehomogenization of f_i, $i = 1, \ldots, n-1$. It follows from Theorem 2.5 that $C_{\ell,m}$ is a s.t.c.i. on the hypersurfaces $g_i = 0$ and $G = {x_1}^{s_1} \cdots {x_n}^{s_n} - x_{n+1}^{\ell} = 0$. So, the ideal of the affine curve $C_{\ell,m}$ contains g_i's and G. Hence the ideal of the projective closure of $C_{\ell,m}$ must contain (at least) f_i's and F, where F is* *the homogenization of G. Now, since $f_1, \ldots, f_{n-1}, F \in I(\overline{C}_{\ell,m})$, we always have $\overline{C}_{\ell,m} \subseteq Z(f_1, \ldots, f_{n-1}, F)$.*

4.1.1 Special Extensions of Arbitrary Monomial Curves

In this section we assume that m is a multiple of m_n, i.e. $m = s_n m_n$ where s_n is a positive integer. Note that $(s_1, \ldots, s_{n-1}) = (0, \ldots, 0)$ and $\delta(m) = s_n$ in this case. This special choice enable us to prove the following

Theorem 4.2 *Let $\overline{C} \subset \mathbb{P}^n$ be a s.t.c.i. on the hypersurfaces $f_1 = \cdots = f_{n-1} = 0$, $gcd(\ell, s_n m_n) = 1$ and $s_n > \ell$. Then, the nice extensions $\overline{C}_{\ell,s_n m_n} \subset \mathbb{P}^{n+1}$ are s.t.c.i. on $f_1 = \cdots = f_{n-1} = F = 0$ where $F = x_n^{s_n} - x_0^{s_n-\ell} x_{n+1}^{\ell}$.*

Proof: The fact that these nice extensions are s.t.c.i. can be seen easily by [77, Theorem 3.4] taking $b_1 = m_1, \ldots, b_{n-1} = m_{n-1}$, $d = m_n$ and $k = (s_n - \ell)m_n$. In addition to this, we provide here the equation of the binomial hypersurface $F = 0$ on which these extensions lie as s.t.c.i. monomial curves.

Since $\overline{C}_{\ell,s_n m_n} \subseteq Z(f_1, \ldots, f_{n-1}, F)$, we need to get the converse inclusion. Take a point $P = (p_0, \ldots, p_n, p_{n+1}) \in Z(f_1, \ldots, f_{n-1}, F)$. Then, since $f_i \in K[x_0, \ldots, x_n]$, we have $f_i(P) = f_i(p_0, \ldots, p_n) = 0$, for all $i = 1, \ldots, n-1$. Since $Z(f_1, \ldots, f_{n-1}) = \overline{C}$ in $\mathbb{P}^n$ by assumption, the last observation implies that

$$(p_0, \ldots, p_n) = (u^{m_n}, u^{m_n - m_1} v^{m_1}, \ldots, u^{m_n - m_{n-1}} v^{m_{n-1}}, v^{m_n}).$$

If $p_0 = 0$ then $u = 0$, yielding that $(p_0, \ldots, p_{n-1}, p_n) = (0, \ldots, 0, p_n)$. Since $s_n > \ell$, we have also $p_n = 0$, by $F(0, \ldots, 0, p_n, p_{n+1}) = {p_n}^{s_n} - p_0^{s_n-\ell} p_{n+1}^{\ell} = 0$. So we observe that $(p_0, \ldots, p_n, p_{n+1}) = (0, \ldots, 0, 1)$ which is on the curve $\overline{C}_{\ell,s_n m_n}$. If $p_0 = 1$ then $(1, p_1, \ldots, p_n, p_{n+1}) \in Z(g_1, \ldots, g_{n-1}, G)$ by the assumption, where g_i and G are polynomials defined in Remark 4.1. Since $C_{\ell,s_n m_n}$ is a s.t.c.i. on the hypersurfaces $g_1 = \cdots = g_{n-1} = 0$ and $G = 0$ it follows that $(1, p_1, \ldots, p_n, p_{n+1}) \in C_{\ell,s_n m_n} \subset \overline{C}_{\ell,s_n m_n}$. $\square$

Since Arithmetically Cohen-Macaulay monomial curves are s.t.c.i. in $\mathbb{P}^3$ (see [63]), we get the following corollary as a consequence of Theorem 4.2.

Corollary 4.3 *Let $\overline{C}(m_1, m_2, m_3)$ be an Arithmetically Cohen-Macaulay monomial curve in $\mathbb{P}^3$. Let $m = s_3 m_3$, $gcd(\ell, m) = 1$ and $\delta(m) = s_3 > \ell$. Then the nice extensions $\overline{C}_{\ell, s_3 m_3} = \overline{C}(\ell m_1, \ell m_2, \ell m_3, s_3 m_3)$ are all s.t.c.i. in $\mathbb{P}^4$.* □

Remark 4.4 *There are very few examples of s.t.c.i. monomial curves in $\mathbb{P}^n$, where $n > 3$. We know that rational normal curve $\overline{C}(1, 2, \ldots, n)$ is a s.t.c.i. in $\mathbb{P}^n$, for any $n > 0$, (see [62, 77]). Applying Theorem 4.2 to $\overline{C}(1, 2, \ldots, n) \subset \mathbb{P}^n$, we can produce infinitely many new examples of s.t.c.i. monomial curves in $\mathbb{P}^{n+1}$:*

Corollary 4.5 *For all positive integers ℓ, n and s with $gcd(\ell, sn) = 1$, the monomial curves $\overline{C}(\ell, 2\ell, \ldots, n\ell, sn) \subset \mathbb{P}^{n+1}$ are s.t.c.i.*

Proof: Let $m = sn$. Clearly $\delta(m) = s$. If $s \leq \ell$, then the monomial curves $\overline{C}_{\ell,m} = \overline{C}(\ell, 2\ell, \ldots, n\ell, sn) \subset \mathbb{P}^{n+1}$ are bad extensions of $\overline{C}(1, 2, \ldots, n) \subset \mathbb{P}^n$. Hence they are s.t.c.i. by Corollary 2.10. If $s > \ell$, then these curves are nice extensions of $\overline{C}(1, 2, \ldots, n) \subset \mathbb{P}^n$. Therefore they are s.t.c.i. by Theorem 4.2. □

In [52], all (ideal theoretic) complete intersection (i.t.c.i.) lattice ideals are characterized by gluing semigroups. But, for a given projective monomial curve it is not easy to find two subsemigroups whose ideals are complete intersection. So, as another application of Theorem 4.2 we can produce infinitely many i.t.c.i. monomial curves:

Proposition 4.6 *If $\overline{C} \subset \mathbb{P}^n$ is an i.t.c.i., then the nice extensions $\overline{C}_{\ell, s_n m_n} \subset \mathbb{P}^{n+1}$ are i.t.c.i. for all positive integers ℓ and s_n with $s_n > \ell$, $gcd(\ell, s_n m_n) = 1$.*

Proof: Since $\overline{C}$ is a s.t.c.i. on the binomial hypersurfaces $f_1 = \cdots = f_{n-1} = 0$, it follows from Theorem 4.2 that $\overline{C}_{\ell, s_n m_n}$ is a s.t.c.i. on $f_1 = \cdots = f_{n-1} = 0$ and $F(x_0, \ldots, x_{n+1}) = x_n^{s_n} - x_0^{s_n - \ell} x_{n+1}^{\ell} = 0$. Since these are all binomial, the monomial curves $\overline{C}_{\ell, s_n m_n}$ are i.t.c.i. on the same hypersurfaces, by [9, Theorem 4]. □

Corollary 4.7 *The monomial curves $\overline{C}(\ell m_1, \ell m_2, s_2 m_2)$ are i.t.c.i. in $\mathbb{P}^3$, for all positive integers m_1, m_2, ℓ and s_2 with $s_2 > \ell$, $gcd(\ell, s_2 m_2) = 1$.*

Proof: Let $m = s_2 m_2$. Then $\delta(m) = s_2$ and $\overline{C}_{\ell,m} = \overline{C}(\ell m_1, \ell m_2, s_2 m_2)$ is a nice extension of $\overline{C}(m_1, m_2)$, by the assumption $s_2 > \ell$. Since $\overline{C}(m_1, m_2)$ is an i.t.c.i. on $x_1^{m_2} - x_0^{m_2-m_1} x_2^{m_1} = 0$, it follows from Proposition 4.6 that the nice extensions $\overline{C}(\ell m_1, \ell m_2, s_2 m_2)$ are i.t.c.i. on $x_1^{m_2} - x_0^{m_2-m_1} x_2^{m_1} = 0$ and $x_2^{s_2} - x_0^{s_2-\ell} x_3^{\ell} = 0$. □

To produce infinitely many examples of i.t.c.i. curves, our method starts from just one i.t.c.i. curve, whereas semigroup gluing method produces only one example starting from one i.t.c.i.. The following example illustrates this point.

Example 4.8 *From Corollary 4.7, we know that $\overline{C}(1, 2, 4)$ is an i.t.c.i. on*

$$f_1 = x_1^2 - x_0 x_2 = 0 \quad \textit{and} \quad f_2 = x_2^2 - x_0 x_3 = 0.$$

Take two positive integers ℓ and s with $s > \ell$, $gcd(\ell, 4s) = 1$. Then the monomial curves $\overline{C}(\ell, 2\ell, 4\ell, 4s) \subset \mathbb{P}^4$ are nice extensions of $\overline{C}(1, 2, 4) \subset \mathbb{P}^3$. Thus, by Proposition 4.6, the monomial curves $\overline{C}(\ell, 2\ell, 4\ell, 4s)$ are i.t.c.i. on

$$f_1 = x_1^2 - x_0 x_2 = 0, \; f_2 = x_2^2 - x_0 x_3 = 0 \quad \textit{and} \quad F = x_3^s - x_0^{s-\ell} x_4^{\ell} = 0.$$

The nice extensions $\overline{C}(\ell, 2\ell, 4\ell, 4s)$ can also be obtained by gluing subsemigroups generated by $T_1 = \{(4s - \ell, \ell)\}$ and $T_2 = \{(4s, 0), (4s - 2\ell, 2\ell), (4s - 4\ell, 4\ell), (0, 4s)\}$. But, in this case one has to know that $\overline{C}(\ell, 2\ell, 2s)$ is an i.t.c.i. for each ℓ and s. In other words, starting with the fact that $\overline{C}(1, 2, 4)$ is an i.t.c.i., gluing method can only produce $\overline{C}(1, 2, 4, 8)$ as an i.t.c.i. monomial curve.

4.1.2 Arbitrary Extensions of Special Monomial Curves

Assume now that m is not a multiple of m_n, i.e. $(s_1, \ldots, s_{n-1}) \neq (0, \ldots, 0)$. Recall that we choose $s_1, \ldots, s_n$ in the representation of $m = s_1 m_1 + \cdots + s_n m_n$ in such a way that $s_1 + \cdots + s_n$ is minimum, i.e. $s_1 + \cdots + s_n = \delta(m)$. First we prove a lemma where no restriction on the f_i is required.

Lemma 4.9 *Let $\overline{C} \subset \mathbb{P}^n$ be a s.t.c.i. on $f_1 = \cdots = f_{n-1} = 0$ and $\delta(m) > \ell$. Then, $Z(f_1, \ldots, f_{n-1}, F) = \overline{C}_{\ell,m} \cup L \subset \mathbb{P}^{n+1}$, where $F = {x_1}^{s_1} \cdots {x_n}^{s_n} - x_0^{\delta(m)-\ell} x_{n+1}^{\ell}$ and L is the line $x_0 = \cdots = x_{n-1} = 0$.*

Proof: We first prove $\overline{C}_{\ell,m} \bigcup L \subseteq Z(f_1, \ldots, f_{n-1}, F)$. By the light of Remark 4.1, it is sufficient to see that $L \subseteq Z(f_1, \ldots, f_{n-1}, F)$. For this, we take a point $P = (p_0, \ldots, p_{n+1})$ on the line L, i.e., $P = (0, \ldots, 0, p_n, p_{n+1})$. Since $(s_1, \ldots, s_{n-1}) \neq (0, \ldots, 0)$ and $\delta(m) > \ell$, we see that $F(P) = 0$. Letting $v \in K$ be any m_n-th root of p_n, we get $(0, \ldots, 0, p_n) = (0, \ldots, 0, v^{m_n}) \in \overline{C} = Z(f_1, \ldots, f_{n-1})$. Since the polynomials f_i are in $K[x_0, \ldots, x_n]$, it follows that $f_i(P) = f_i(0, \ldots, 0, p_n) = 0$, for all $i = 1, \ldots, n-1$. Thus $P \in Z(f_1, \ldots, f_{n-1}, F)$.

For the converse inclusion, take $P = (p_0, \ldots, p_n, p_{n+1}) \in Z(f_1, \ldots, f_{n-1}, F)$. Then, for all $i = 0, \ldots, n-1$, we get $f_i(p_0, \ldots, p_n) = f_i(P) = 0$ implying that

$$(p_0, \ldots, p_n) = (u^{m_n}, u^{m_n - m_1} v^{m_1}, \ldots, u^{m_n - m_{n-1}} v^{m_{n-1}}, v^{m_n}).$$

If $p_0 = 0$ then $u = 0$, yielding that $(p_0, \ldots, p_n) = (0, \ldots, 0, p_n)$. Thus, we get $P = (p_0, \ldots, p_n, p_{n+1}) = (0, \ldots, 0, p_n, p_{n+1}) \in L$. If $p_0 = 1$ then by assumption we know that $P = (1, p_1, \ldots, p_n, p_{n+1}) \in Z(g_1, \ldots, g_{n-1}, G)$. Since $C_{\ell,m}$ is a s.t.c.i. on the hypersurfaces $g_1 = \cdots = g_{n-1} = 0$ and $G = 0$ it follows that $P = (1, p_1, \ldots, p_n, p_{n+1}) \in C_{\ell,m} \subset \overline{C}_{\ell,m}$. □

To get rid of L in the intersection of the hypersurfaces $f_1 = \cdots = f_{n-1} = 0$ and $F = 0$, we modify the $F = {x_1}^{s_1} \cdots {x_n}^{s_n} - x_0^{\delta(m)-\ell} x_{n+1}^{\ell}$ of the Lemma 4.9, as in the work of Bresinsky (see [11]), for some special choice of $f_1, \ldots, f_{n-1}$. In this way we construct a new polynomial F^* from F such that $Z(f_1, \ldots, f_{n-1}, F^*) = \overline{C}_{\ell,m}$, where F^* is a polynomial of the form

$$F^* = x_n^{\alpha} + x_0^{\beta} H(x_0, \ldots, x_{n+1}),$$

where β is a positive integer.

Note that when $x_0 = 0$, the vanishing of F^* implies that $x_n = 0$. It follows from the last part of the proof of Lemma 4.9 that this property of F^* ensures that we have a point at infinity, in the intersection of $f_1 = \cdots = f_{n-1} = 0$ and $F^* = 0$, instead of a line.

The construction of F^* can be described as follows. We first assume that $f_i = x_i^{a_i} - x_0^{a_i - b_i} x_n^{b_i} = 0$, where $a_i > b_i$ are positive integers, for all $i = 1, \ldots, n-1$. Let $p = a_1 \cdots a_{n-1}$ and $p_i = \frac{b_i}{a_i} p$, for $i = 1, \ldots, n-1$. Take the p-th power of F and for every occurrence of $x_i^{a_i}$ substitute $x_0^{a_i - b_i} x_n^{b_i}$, for all $i = 1, \ldots, n-1$. Then we have

$$\begin{aligned} F^p &= x_0^{\gamma} x_n^{\alpha} + x_0^{\delta(m)-\ell} H(x_0, \ldots, x_{n+1}) \;\; mod(f_1, \ldots, f_{n-1}) \\ &= x_0^{\gamma}[x_n^{\alpha} + x_0^{\delta(m)-\ell-\gamma} H(x_0, \ldots, x_{n+1})] \;\; mod(f_1, \ldots, f_{n-1}) \end{aligned}$$

where $\gamma = \sum_{i=1}^{n-1}(p - p_i)s_i$, $\alpha = ps_n + \sum_{i=1}^{n-1} p_i s_i$ and H is a polynomial. Letting

$$F^*(x_0, \ldots, x_{n+1}) = x_n^{\alpha} + x_0^{\delta(m)-\ell-\gamma} H(x_0, \ldots, x_{n+1})$$

we observe that

$$F^p(x_0, \ldots, x_{n+1}) = x_0^{\gamma} F^*(x_0, \ldots, x_{n+1}) \;\; mod(f_1, \ldots, f_{n-1}). \tag{4.1}$$

Recall that m is an element of the numerical semigroup generated by $m_1, \ldots, m_n$, i.e. $m = s_1 m_1 + \cdots + s_n m_n$ with $s_1 + \cdots + s_n = \delta(m)$. If m is large enough that $s_n > \ell + \sum_{i=1}^{n-1}(p - p_i - 1)s_i$ (or equivalently $\delta(m) - \ell - \gamma > 0$) then F^* is the required polynomial. (Otherwise, F^* may not be a polynomial.) Hence we conclude the following

Theorem 4.10 *Let p, p_i, f_i and F^* be as above. Assume that m is chosen so that $s_n > \ell + \sum_{i=1}^{n-1}(p - p_i - 1)s_i$. Then, for all $\ell < \delta(m)$ with $gcd(\ell, m) = 1$, the nice extensions $\overline{C}_{\ell,m} \subset \mathbb{P}^{n+1}$ are s.t.c.i. on $f_1 = \cdots = f_{n-1} = 0$ and $F^* = 0$.*

Proof: We will show that $\overline{C}_{\ell,m}$ is a s.t.c.i. on $f_1 = \cdots = f_{n-1} = 0$ and $F^* = 0$. To do this, take a point $P = (p_0, \ldots, p_{n+1}) \in \overline{C}_{\ell,m}$. Then, $F(P) = 0$ and $f_i(P) = 0$, for all $i = 1, \ldots, n-1$, since $Z(f_1, \ldots, f_{n-1}, F) = \overline{C}_{\ell,m} \bigcup L$, by Lemma 4.9. From equation (4.1) it follows that $F^*(P) = 0$ or $p_0 = 0$. Since P is a point on the monomial curve $\overline{C}_{\ell,m}$, it can be parameterized as follows:

$$(u^m, u^{m-\ell m_1} v^{\ell m_1}, \ldots, u^{m-\ell m_n} v^{\ell m_n}, v^m)$$

So if $p_0 = 0$, we get $u = 0$ and thus $p_i = 0$, for all $i = 1, \ldots, n$. Therefore $P = (0, \ldots, 0, 1)$ and hence $F^*(P) = 0$ in any case.

Conversely, let $P = (p_0, \ldots, p_{n+1}) \in Z(f_1, \ldots, f_{n-1}, F^*)$. If $p_0 = 0$, then $p_i = 0$ by $f_i(P) = 0$, for all $i = 1, \ldots, n-1$. Since $\delta(m) - \ell - \gamma > 0$, we have $p_n = 0$ by $F^*(P) = 0$. Thus $P = (0, \ldots, 0, 1)$ which is always on the curve $\overline{C}_{\ell,m}$. If $p_0 = 1$ then C is a s.t.c.i. on the hypersurfaces given by $g_i = x_i^{a_i} - x_{i+1}^{b_i} = 0$, for $i = 1, \ldots, n-1$, by the assumption. Hence, Theorem 2.5 implies that $C_{\ell,m}$ is a s.t.c.i. on $g_1 = \cdots = g_{n-1} = 0$ and $G = {x_1}^{s_1} \cdots {x_n}^{s_n} - x_{n+1}^{\ell} = 0$. Thus $P = (1, p_1, \ldots, p_{n+1}) \in C_{\ell,m} \subset \overline{C}_{\ell,m}$. $\square$

Remark 4.11 *The* nice *extensions in Theorem 4.10 can also be shown to be s.t.c.i. by using [77, Theorem 3.4]. But to show that the hypotheses of [77, Theorem 3.4] are satisfied by these extensions is much more difficult than the proof here. As a byproduct we also constructed here the hypersurface* $F^* = 0$ *on which these* nice *extensions are s.t.c.i.*

Example 4.12 *We start with* $\overline{C} = \overline{C}(3, 4, 6) \subset \mathbb{P}^3$. *Let* $\ell = 1$ *and* $m = 6s + 7$, *for some positive integer* s. *Then* $\delta(m) = s + 2$, $s_1 = s_2 = 1$ *and* $s_3 = s$. *Thus we get the nice extensions* $\overline{C}_{1,6s+7} = \overline{C}(3, 4, 6, 6s + 7) \subset \mathbb{P}^4$. *Since* $\Delta_1 = gcd(4, 6, 6s + 7) = 1$, $\Delta_2 = gcd(3, 6, 6s + 7) = 1$ *and* $\Delta_3 = gcd(3, 4, 6s + 7) = 1$ *it follows from Corollary 2.8 that these curves can not be obtained by gluing. Using the software Macaulay [30], it is easy to see that the ideal of* $\overline{C}_{1,6s+7}$ *is minimally generated by the polynomials*

$$\begin{aligned}
f_1 &= x_1^2 - x_0 x_3, \\
f_2 &= x_2^3 - x_0 x_3^2, \\
f_3 &= x_3^{s+3} - x_0^{s-1} x_1 x_2^2 x_4 \\
f_4 &= x_2 x_3^{s+1} - x_0^s x_1 x_4, \\
f_5 &= x_1 x_3^{s+2} - x_0^s x_2^2 x_4 \\
F &= x_1 x_2 x_3^s - x_0^{s+1} x_4.
\end{aligned}$$

Since $\overline{C}(3, 4, 6) \subset \mathbb{P}^3$ *is a s.t.c.i. on the surfaces* $f_1 = 0$ *and* $f_2 = 0$, *it follows* *from Theorem 4.10 that $\overline{C}_{1,6s+7}$ is a s.t.c.i. on $f_1 = 0$, $f_2 = 0$ and*

$$F^* = x_3^{6s+7} - 6x_0^{s-1}x_1x_2^2x_3^{5s+4}x_4 + 15x_0^{2s}x_2x_3^{4s+4}x_4^2 - 20x_0^{3s}x_1x_3^{3s+3}x_4^3 +$$

$$+15x_0^{4s}x_2^2x_3^{2s+1}x_4^4 - 6x_0^{5s}x_1x_2x_3^sx_4^5 + x_0^{6s+1}x_4^6 = 0$$

provided that $s > 2$.

Recall that our method starts with a monomial curve $\overline{C} = Z(f_1, \ldots, f_{n-1})$ in $\mathbb{P}^n$ and produces infinitely many nice extensions $\overline{C}_{\ell,m} = Z(f_1, \ldots, f_{n-1}, F^*)$ in $\mathbb{P}^{n+1}$. Since the construction of F^* depends on the choice of $f_1, \ldots, f_{n-1}$, it is possible to start with another curve $\overline{C} = Z(f_1, \ldots, f_{n-1})$ in $\mathbb{P}^n$ and obtain new families of nice extensions. Now we provide two examples of this sort. For instance, if we assume that $\overline{C}$ is a s.t.c.i. on the hypersurfaces $f_i = x_i^{a_i} - x_0^{a_i-b_i}x_{i+1}^{b_i} = 0$, where $a_i > b_i$ are positive integers, $i = 1, \ldots, n-1$, then under some suitable conditions we obtain other families of s.t.c.i. nice extensions. Let $p = a_1 \cdots a_{n-1}$, $q_0 = b_1 \cdots b_{n-1}$ and $q_i = a_1 \cdots a_i b_{i+1} \cdots b_{n-1}$, $i = 1, \ldots, n-2$. The first variation is the following

Theorem 4.13 *Let $p, q_0, \ldots, q_{n-2}$ be as above. For all m which give rise to $s_n > \ell + \sum_{i=0}^{n-2}(p - q_i - 1)s_{i+1}$ and for all ℓ with $\ell < \delta(m)$ and $gcd(\ell, m) = 1$, the nice extensions $\overline{C}_{\ell,m} \subset \mathbb{P}^{n+1}$ are s.t.c.i. on $f_1 = \cdots = f_{n-1} = F^* = 0$.*

Proof: Let $F = {x_1}^{s_1} \ldots {x_n}^{s_n} - x_0^{\delta(m)-\ell}x_{n+1}^{\ell}$. Taking the p-th power and replacing $x_i^{a_i}$ by $x_0^{a_i-b_i}x_{i+1}^{b_i}$ for each $i = 1, \ldots, n-1$ we get the following

$$\begin{aligned} F^p &= x_0^{\gamma}x_n^{\alpha} + x_0^{\delta(m)-\ell}H(x_0, \ldots, x_{n+1}) \;\; mod(f_1, \ldots, f_{n-1}) \\ &= x_0^{\gamma}[x_n^{\alpha} + x_0^{\delta(m)-\ell-\gamma}H(x_0, \ldots, x_{n+1})] \;\; mod(f_1, \ldots, f_{n-1}) \end{aligned}$$

where $\gamma = \sum_{i=0}^{n-2}(p - q_i)s_{i+1}$, $\alpha = ps_n + \sum_{i=0}^{n-2} q_i s_{i+1}$ and H is a polynomial. Letting

$$F^*(x_0, \ldots, x_{n+1}) = x_n^{\alpha} + x_0^{\delta(m)-\ell-\gamma}H(x_0, \ldots, x_{n+1})$$

we observe that

$$F^p(x_0, \ldots, x_{n+1}) = x_0^{\gamma}F^*(x_0, \ldots, x_{n+1}) \;\; mod(f_1, \ldots, f_{n-1}).$$

The proof of the claim that $\overline{C}_{\ell,m}$ is a s.t.c.i. on $f_1 = \cdots = f_{n-1} = F^* = 0$ can be done as in the proof of the Theorem 4.10. $\square$

Now, we give another variation where $m = s_i m_i + s_j m_j$, for $i, j \in \{1, \ldots, n\}$. For the notational convenience we take $i = 1$ and $j = n$.

Theorem 4.14 *Let $\overline{C} \subset \mathbb{P}^n$ be a s.t.c.i. on the hypersurfaces given by*

$$\begin{aligned} f_1 &= x_1^a - x_0^{a-b} x_n^b = 0 \\ f_i &= x_i^{a_i} + x_0^{b_i} A(x_1, \ldots, x_n) + x_1^{c_i} B(x_2, \ldots, x_n) = 0, \end{aligned}$$

where a, b, $a-b$, a_i, b_i, and c_i are positive integers, for $i = 2, \ldots, n-1$, A and B are some polynomials. For all m which give rise to $s_n > \ell + (a-b-1)s_1$ and for all ℓ with $\ell < \delta(m)$ and $gcd(\ell, m) = 1$, the nice extensions $\overline{C}_{\ell,m} \subset \mathbb{P}^{n+1}$ are s.t.c.i. on $f_1 = \cdots = f_{n-1} = F^* = 0$.

Proof: Let $F = {x_1}^{s_1} {x_n}^{s_n} - x_0^{s_1+s_n-\ell} x_{n+1}^{\ell}$. Then it is easy to see the following

$$F^a = x_0^{(a-b)s_1} F^*(x_0, \ldots, x_{n+1}) \ \ (mod\ f_1) \quad \text{where}$$

$$F^* = x_n^{bs_1+as_n} + x_0^{(1+b-a)s_1+s_n-\ell} \sum_{k=1}^{a} (-1)^k \binom{a}{k} (x_1^{s_1} {x_n}^{s_n})^{a-k} x_0^{(s_1+s_n-\ell)(k-1)} x_{n+1}^{k\ell}.$$

The proof of the claim that $\overline{C}_{\ell,m}$ is a s.t.c.i. on $f_1 = \cdots = f_{n-1} = F^* = 0$ can be done as in the proof of the Theorem 4.10. $\square$

Example 4.15 *Consider the monomial curve $\overline{C}(3, 5, 9, 9s + 5) \subset \mathbb{P}^4$, for all $s \geq 2$. Since $gcd(5, 9, 9s+5) = 1$, $gcd(3, 9, 9s+5) = 1$ and $gcd(3, 5, 9s+5) = 1$ it follows from Corollary 2.8 that these curves can not be obtained by gluing. Using the software Macaulay [30], it is easy to see that the ideal of $\overline{C}(3, 5, 9, 9s + 5)$ is minimally generated by the polynomials*

$$f_1 = x_1^3 - x_0^2 x_3, f_2 = x_2^3 - x_1^2 x_3, f_3 = x_3^{s+2} - x_0^{s-2} x_1 x_2^2 x_4, f_4 = x_2 x_3^s - x_0^s x_4$$

and $F = x_1^2 x_3^{s-1} - x_0^s x_2^2 x_4$. Since $\overline{C}(3, 5, 9) \subset \mathbb{P}^3$ is a s.t.c.i. on the surfaces $f_1 = 0$ and $f_2 = 0$, it follows from Theorem 4.14 that $\overline{C}_{1,9s+5} = \overline{C}(3, 5, 9, 9s + 5)$ is a s.t.c.i. on $f_1 = 0$, $f_2 = 0$ and

$$F^* = x_3^{3s+4} - 3x_0^{s-2} x_1^2 x_2^2 x_3^{2s+2} x_4 + 3x_0^{2s-2} x_1 x_2^4 x_3^{s+1} x_4^2 - x_0^{3s-2} x_2^6 x_4^3 = 0.$$

Example 4.16 *By Corollary 4.7, we know that $\overline{C}(1,2,4) \subset \mathbb{P}^3$ is an i.t.c.i. on $f_1 = x_1^2 - x_0x_2 = 0$ and $f_2 = x_2^2 - x_0x_3 = 0$. In this example, we show that the monomial curve $\overline{C}(1,2,4,m) \subset \mathbb{P}^4$ is a s.t.c.i. for any $m \neq 5, 7$. Clearly m is 0, 1, 2 or 3 (mod 4). The case $m = 4s$ is investigated in Example 4.8. In the case of $m = 4s+1$, we have the monomial curve $\overline{C}(1,2,4,4s+1) \subset \mathbb{P}^4$ whose ideal is generated by the following set of generators*

$$f_1, f_2, f_3 = x_2x_3^s - x_0^{s-1}x_1x_4, f_4 = x_3^{s+1} - x_0^{s-2}x_1x_2x_4, F = x_1x_3^s - x_0^sx_4.$$

Since $m = 4s+1$, this means that $s_1 = 1$, $s_2 = 0$ and $s_3 = s$ in Theorem 4.13. In the theorem we assume that $s_3 = s > \ell + 2s_1 + s_2 = 3$ but this is not sharp. Indeed, the construction of F^* work if $s > 1$. The construction is as follows:

$$F^4 = (x_1x_3^s - x_0^sx_4)^4 = x_1^4x_3^{4s} - 4x_1^3x_3^{3s}x_0^sx_4 + 6x_1^2x_3^{2s}x_0^{2s}x_4^2 - 4x_1x_3^sx_0^{3s}x_4^3 + x_0^{4s}x_4^4.$$

Since $x_1^2 = x_0x_2 \mod(f_1)$ and $x_2^2 = x_0x_3 \mod(f_2)$, it follows that we have $x_1^4 = x_0^2x_2^2 = x_0^3x_3 \mod(f_1, f_2)$. Thus, we get $F^4 = x_0^3(F^*) \mod(f_1, f_2)$, where

$$F^* = x_3^{4s+1} - 4x_0^{s-2}x_1x_2x_3^{3s}x_4 + 6x_0^{2s-2}x_2x_3^{2s}x_4^2 - 4x_0^{3s-3}x_1x_3^sx_4^3 + x_0^{4s-3}x_4^4.$$

Thus, the curve $\overline{C}(1,2,4,4s+1) \subset \mathbb{P}^4$ is a s.t.c.i. on $f_1 = 0$, $f_2 = 0$ and $F^* = 0$. In the case where $s = 1$, F^* is not a polynomial since $x_0^{s-2}x_1x_2x_3^{3s}x_4$ is not a monomial. That's why our method does not apply here.

If $m = 4s+2$, we have the monomial curve $\overline{C}(1,2,4,4s+2) \subset \mathbb{P}^4$ whose ideal is generated by the following set of generators

$$f_1, f_2, f_3 = x_3^{s+1} - x_0^{s-1}x_2x_4, F = x_2x_3^s - x_0^sx_4.$$

In this case we take $s_1 = 0$, $s_2 = 1$ and $s_3 = s > 2$ to apply Theorem 4.13, which yields $F^4 = x_0^2(F^*) \mod(f_1, f_2)$, where

$$F^* = (x_3^{4s+2} - 2x_0^{s-1}x_2x_3^sx_4 + x_0^{2s-1}x_4^2)^2.$$

Thus, the curve $\overline{C}(1,2,4,4s+2) \subset \mathbb{P}^4$ is a s.t.c.i. on $f_1 = 0$, $f_2 = 0$ and $F^* = 0$. Indeed, we could apply Theorem 4.14 here with $s > 1$ and in this case we get a quadric G^* instead of a quartic F^* above. We take 2nd power of F and mode it by f_2 to get:

$$F^2 = x_2^2x_3^{2s} - 2x_0^sx_2x_3^sx_4 + x_0^{2s}x_4^2 = x_0G^* \quad \textit{where}$$

$$G^* = x_3^{2s+1} - 2x_0^{s-1}x_2x_3^sx_4 + x_0^{2s-1}x_4^2.$$

Note that $F^* = (G^*)^2$ *and* $\overline{C}(1,2,4,4s+2) \subset \mathbb{P}^4$ *is a s.t.c.i. on* $f_1 = 0$, $f_2 = 0$ *and* $G^* = 0$.

If $m = 4s+3$, *we have the monomial curve* $\overline{C}(1,2,4,4s+3) \subset \mathbb{P}^4$ *whose ideal is generated by the following set of generators*

$$f_1, f_2, f_3 = x_3^{s+1} - x_0^{s-1}x_1x_4, F = x_1x_2x_3^s - x_0^{s+1}x_4.$$

Now, we need to take $s_1 = 1$, $s_2 = 1$ *and* $s_3 = s > 4$ *to apply Theorem 4.13, but the same happens to be true for any positive integer* s. *As before, we have the following relation* $F^4 = x_0^5(F^*)\ \ mod(f_1, f_2)$, *where*

$$F^* = x_3^{4s+3} - 4x_0^{s-1}x_1x_3^{3s+2}x_4 + 6x_0^{2s-1}x_2x_3^{2s+1}x_4^2 - 4x_0^{3s-2}x_1x_2x_3^sx_4^3 + x_0^{4s-1}x_4^4.$$

Thus, the curve $\overline{C}(1,2,4,4s+1) \subset \mathbb{P}^4$ *is a s.t.c.i. on* $f_1 = 0$, $f_2 = 0$ *and* $F^* = 0$. *So the missing integers are* $m = 5, 6, 7$ *corresponding to* $s = 1$.

When $m = 6$, *we use Theorem 4.2 with* $\ell = 2$, $s_3m_3 = 1$ *and the fact that* $\overline{C}(3,2,1)$ *is a s.t.c.i. on* $x_3^2 = x_0x_2$ *and* $x_2^3 - 2x_1x_2x_3 + x_0x_1^2 = 0$. *So* $\overline{C}(6,4,2,1)$ *is a s.t.c.i. on* $x_3^2 = x_0x_2$, $x_2^3 - 2x_1x_2x_3 + x_0x_1^2 = 0$ *and* $x_4^2 = x_0x_3$ *implying that* $\overline{C}(1,2,4,6)$ *is a s.t.c.i. on* $x_2^2 = x_0x_3$, $x_3^3 - 2x_2x_3x_4 + x_0x_4^2 = 0$ *and* $x_1^2 = x_0x_2$.

Thus the only open cases that the technique of this thesis does not apply are $m = 5$ *and* 7 *for this example.*

Chapter 5

Hilbert Function of Monomial Curves

In this chapter, we study the Hilbert functions of local rings associated to monomial curves. Our aim is to obtain large families of one dimensional local rings with arbitrary embedding dimension whose Hilbert function is non-decreasing. This will be achieved by producing affine monomial curves whose tangent cones are Cohen-Macaulay by using the technique of gluing numerical semigroups. The Cohen-Macaulayness of the tangent cones of monomial curves has been studied by many authors, see [2], [4], [15], [28], [49], [50], [61] and [67]. To check the Cohen-Macaulayness, we first present an easy and efficient criterion by using the standard basis theory. This new criterion refines the given one in the literature. We use this criterion and the technique of gluing to obtain infinitely many new families of monomial curves in arbitrary dimensions with Cohen-Macaulay tangent cones. In this way, we generalize the results in [2] and [4] given for nice extensions, which are in fact special types of gluings. In doing this, we also give the definition of a nice gluing which is a generalization of a nice extension defined in [4]. The content of this chapter is a fruit of our joint work with Feza Arslan and Pınar Mete, see also [5]. We encourage the reader to consult [3] for fundamental facts about tangent cone of a monomial curve and its Cohen-Macaulayness and to [46] for their Hilbert functions.

Let S be a polynomial ring $K[x_1, \dots, x_k]$ over a field K. If M is a finitely generated $\mathbb{N}$-graded S-module, i.e. $M = \bigoplus_{r \in \mathbb{N}} M_r$, then the Hilbert function of M is defined to be $H_M(r) = dim_K M_r$, where the graded modules M_r are finite dimensional vector spaces over K. The Hilbert series $HP_M(y)$ of M is defined to be the power series $\sum_{r \in \mathbb{N}} H_M(r) y^r$. For example, the Hilbert function and Hilbert series of S itself are given by the following combinatorial formulas:

$$H_S(r) = \binom{k-1+r}{k-1} \quad \text{and} \quad HP_S(y) = \sum_{r \in \mathbb{N}} \binom{k-1+r}{k-1} y^r.$$

Let $C = C(n_1, \dots, n_k)$ be a monomial curve corresponding to the numerical semigroup $< n_1, \dots, n_k >$ minimally generated by $n_1, \dots, n_k$. It is known that the coordinate ring $K[C]$ of C is isomorphic to the affine semigroup ring $K[t^{n_1}, \dots, t^{n_k}]$. Clearly, $K[t^{n_1}, \dots, t^{n_k}] = \bigoplus_{r \in \mathbb{N}} K[t^r]$ and $dim_K K[t^r] = 1$ if $r \in < n_1, \dots, n_k >$ and $dim_K K[t^r] = 0$ if $r \notin < n_1, \dots, n_k >$. Thus, Hilbert function of the coordinate ring of C takes only two values 0 and 1:

$$H_{K[C]}(r) = 1 \text{ if } r \in < n_1, \dots, n_k > \text{ and } H_{K[C]}(r) = 0 \text{ if } r \notin < n_1, \dots, n_k > .$$

If c is the Frobenius number of the semigroup $< n_1, \dots, n_k >$, i.e. the largest number not belonging to $< n_1, \dots, n_k >$, then Hilbert function is constant ($H_{K[C]}(r) = 1$) for all $r > c$ since in this case $r \in < n_1, \dots, n_k >$. Thus, it is non-decreasing in this case. If $n_1 = 1$, then r is always in the semigroup, and thus $H_{K[C]}(r) = 1$, for any $r \in \mathbb{N}$. But if $n_1 \neq 1$, then there are certainly gaps, i.e. $r \notin < n_1, \dots, n_k >$, for which $H_{K[C]}(r) = 0$. Therefore, in this case, Hilbert function is NOT non-decreasing. For example, if $C = C(3, 5, 7)$, then the numerical semigroup generated minimally by $3, 5, 7$ is

$$< 3, 5, 7 >= \{0, 3, 5, 6, 7, 8, 9, \dots\} \quad \text{and gaps are} \quad \{1, 2, 4\} \quad \text{with} \quad c = 4.$$

Hence, the Hilbert function of the coordinate ring of $C = C(3, 5, 7)$ is given by the following sequence of numbers $H_{K[C]} = \{1, 0, 0, 1, 0, 1, 1, 1, 1, \dots\}$ and clearly decrease at some points. Therefore we can conclude this paragraph by stating that Hilbert function of the coordinate ring of $C(n_1, \dots, n_k)$ is non-decreasing if and only if $n_1 = 1$.

If $(R, \mathbf{m})$ is a local ring with maximal ideal $\mathbf{m}$, then the Hilbert function of R is defined to be the Hilbert function of its *associated graded ring*

$$gr_{\mathbf{m}}(R) = \bigoplus_{r \in \mathbb{N}} \mathbf{m}^r / \mathbf{m}^{r+1}.$$

Therefore,

$$H_R(r) = dim_K(\mathbf{m}^r / \mathbf{m}^{r+1}).$$

If $(R, \mathbf{m})$ is a one dimensional Cohen-Macaulay local ring with embedding dimension $d := H_R(1)$, the following are known about the conjecture of Sally saying that the Hilbert function $H_R(r)$ is non-decreasing:

- $d = 1$, obvious as $H_R(r) = 1$,
- $d = 2$, proved by Matlis (1977) [45],
- $d = 3$, proved by Elias (1993) [21],
- $d = 4$, a counterexample is given by Gupta-Roberts (1983) [29],
- $d \geq 5$, counterexamples for each d are given by Orecchia(1980) [57].

The first counterexamples were the local rings associated to monomial curves. Herzog and Waldi [37] in 1975 were the first who consider the monomial curve $C(30, 35, 42, 47, 148, 153, 157, 169, 181, 193)$ in $\mathbb{A}^{10}$ and its associated local ring $(R, \mathbf{m})$. They show that the Hilbert function of R is NOT non-decreasing by explicitly writing it down:

$$H_R = \{1, 10, \mathbf{9}, 16, 25, \dots\}.$$

Later, Eakin and Sathaye [19] in 1976 took the monomial curve in $\mathbb{A}^{12}$ defined by $C(15, 21, 23, 47, 48, 49, 50, 52, 54, 55, 56, 58)$ and studied its associated local ring $(R, \mathbf{m})$. Hilbert function of R is NOT non-decreasing as it is given by

$$H_R = \{1, 12, \mathbf{11}, 13, 15, \dots\}.$$

5.1 An Effective Criterion for Checking the Cohen-Macaulayness

In this section, we give a refinement of the criterion for checking the Cohen-Macaulayness of the tangent cone of a monomial curve given in [2, Theorem 2.1]. This criterion uses the theorem of Garcia saying that a monomial curve $C = C(n_1, \ldots, n_k)$ with n_1 smallest among the integers $n_1, \ldots, n_k$ has Cohen-Macaulay tangent cone if and only if t^{n_1} is not a zero divisor in $gr_m(k[[t^{n_1}, \ldots, t^{n_k}]])$ (or equivalently, x_1 is not a zero divisor in the ring $K[x_1, \ldots, x_k]/I(C)_*$) [28]. In [2, Theorem 2.1], first the generators of the defining ideal of the tangent cone are computed by a Gröbner basis computation and then from these generators another Gröbner basis is computed in order to check whether x_1 is not a zero divisor. The advantage of this new criterion presented below is that, instead of computing another Gröbner basis after finding the generators of the defining ideal of the tangent cone, it needs only a computation of the standard basis of the generators of the defining ideal of the monomial curve with respect to a special local order. Recall that a local order is a monomial ordering with 1 greater than any other monomial. For the examples and properties of local orderings, see [32].

Lemma 5.1 *Let $< n_1, \ldots, n_k >$ be a numerical semigroup minimally generated by the integers $n_1, \ldots, n_k$ among which n_1 is the smallest. Let $C = C(n_1, \ldots, n_k)$ be the associated monomial curve and $G = \{f_1, \ldots, f_s\}$ be a minimal standard basis of the ideal $I(C) \subset K[x_1, \ldots, x_k]$ with respect to the negative degree reverse lexicographical ordering that makes x_1 the lowest variable. C has Cohen-Macaulay tangent cone at the origin if and only if x_1 does not divide $\mathrm{LM}(f_i)$ for $1 \leq i \leq k$, where $\mathrm{LM}(f_i)$ denotes the leading monomial of a polynomial f_i.*

Proof: Recalling that f_* is the homogeneous summand of the polynomial f of least degree, if x_1 divides $\mathrm{LM}(f_i)$ for some i, then either $f_{i_*} = x_1 m$ or $f_{i_*} = x_1 m + \sum c_i m_i$, where m_i's are monomials having the same degree with $x_1 m$ and c_i's are in K. In the latter case, x_1 must divide each m_i, because we work with the negative degree reverse lexicographical ordering that makes

x_1 the lowest variable. This implies that in both cases $f_{i_*} = x_1 g$ where g is a homogeneous polynomial. Moreover, $g \notin I(C)_*$. If $g \in I(C)_*$, then there exists $f \in I(C)$ such that $f_* = g$ so $\mathrm{LM}(f) = \mathrm{LM}(g)$. Since the ideal generated by the leading monomials of the elements in $I(C)$ (with respect to the negative degree reverse lexicographical ordering which makes x_1 the lowest variable) is equal to the ideal generated by the leading monomials of the elements in G, there exists an $f_j \in G$ such that $\mathrm{LM}(f_j)$ divides $\mathrm{LM}(f) = \mathrm{LM}(g)$ and this contradicts with the minimality of G. Thus, $x_1 g \in I(C)_*$, while $g \notin I(C)_*$, which makes x_1 a zero-divisor in $K[x_1, \ldots, x_k]/I(C)_*$. Hence, the tangent cone of the monomial curve C is not Cohen-Macaulay. Conversely, if $K[x_1, \ldots, x_k]/I(C)_*$ is not Cohen-Macaulay, then x_1 is a zero-divisor in $K[x_1, \ldots, x_k]/I(C)_*$. Thus, $x_1 m \in I(C)_*$, where m is a monomial and $m \notin I(C)_*$. The ideal generated by the leading monomials of the elements in $I(C)$ obviously contains $x_1 m$. Since G is a standard basis, there exists $f_i \in G$ such that $\mathrm{LM}(f_i) = x_1 m'$, where m' divides m and $m' \notin I(C)_*$, because $m \notin I(C)_*$. This completes the proof. □

In this way, checking the Cohen-Macaulayness of the tangent cone of a monomial curve has been just reduced to a computation of a standard basis with respect to the negative degree reverse lexicographical ordering that makes x_1 the lowest variable and checking whether any of the leading monomials of this basis contains x_1.

Example 5.2 *Let C be the monomial curve given by $C = C(6, 7, 15)$. The ideal $I(C)$ is generated by the set $G = \{x_1^5 - x_3^2, x_1x_3 - x_2^3\}$, which has a minimal standard basis with respect to the negative degree reverse lexicographical ordering with $x_2 > x_3 > x_1$ given by the set $G' = \{x_1^5 - x_3^2, x_1x_3 - x_2^3, x_2^3x_3 - x_1^6, x_2^6 - x_1^7\}$. From 5.1, since x_1 divides $\mathrm{LM}(x_1x_3 - x_2^3) = x_1x_3$, the monomial curve C does not have a Cohen-Macaulay tangent cone.*

5.2 Gluing and Cohen-Macaulay Tangent Cones

In this section, we first give the definition of gluing for numerical semigroups.

Definition 5.3 *[65, Lemma 2.2] Let S_1 and S_2 be two numerical semigroups minimally generated by $m_1 < \cdots < m_l$ and $n_1 < \cdots < n_k$ respectively. Let $p = b_1m_1 + \cdots + b_lm_l \in S_1$ and $q = a_1n_1 + \cdots + a_kn_k \in S_2$ be two positive integers satisfying $gcd(p,q) = 1$ with $p \notin \{m_1, \ldots, m_l\}$ and $q \notin \{n_1, \ldots, n_k\}$. The numerical semigroup $S =< qm_1, \ldots, qm_l, pn_1, \ldots, pn_k >$ is called a gluing of the semigroups S_1 and S_2.*

This definition of gluing is different from the one we gave before. In fact S above is the gluing of its subsemigroups qS_1 and pS_2. Since the monomial curve defined by qS_1 is nothing but the one defined by S_1 we prefer to use this definition here.

Thus, the monomial curve $C = C(qm_1, \ldots, qm_l, pn_1, \ldots, pn_k)$ can be interpreted as the gluing of the monomial curves $C_1 = C(m_1, \ldots, m_l)$ and $C_2 = C(n_1, \ldots, n_k)$, if p and q satisfy the conditions in Definition 5.3. Moreoever, if the defining ideals $I(C_1) \subset K[x_1, \ldots, x_l]$ of C_1 and $I(C_2) \subset K[y_1, \ldots, y_k]$ of C_2 are generated by the sets $G_1 = \{f_1, \ldots, f_s\}$ and $G_2 = \{g_1, \ldots, g_t\}$ respectively, then the defining ideal of $I(C) \subset K[x_1, \ldots, x_l, y_1, \ldots, y_k]$ is generated by the set $G = \{f_1, \ldots, f_s, g_1, \ldots, g_t, x_1^{b_1} \ldots x_l^{b_l} - y_1^{a_1} \ldots y_k^{a_k}\}$

We first answer the following question: If C_1 and C_2 have Cohen-Macaulay tangent cones, is the tangent cone of the monomial curve C obtained by gluing these two monomial curves necessarily Cohen-Macaulay? The following example shows that the answer is no.

Example 5.4 *Let C_1 and C_2 be the monomial curves $C_1 = C(5, 12)$ and $C_2 = C(7, 8)$. Obviously, they have Cohen-Macaulay tangent cones. By a gluing of C_1 and C_2, we obtain the monomial curve $C = C(21.5, 21.12, 17.7, 17.8)$. The ideal $I(C)$ is generated by the set $G = \{x_1^{12} - x_2^5, y_1^8 - y_2^7, x_1x_2 - y_1^3\}$, which has a minimal standard basis with respect to the negative degree reverse lexicographical ordering with $x_2 > y_2 > y_1 > x_1$ given by the set $G' = \{x_1x_2 - y_1^3, x_2^5 - x_1^{12}, y_1^{15} - x_1^{17}, y_2^7 - y_1^8, x_2^4y_1^3 - x_1^{13}, x_2^3y_1^6 - x_1^{14}, x_2^2y_1^9 - x_1^{15}, x_2y_1^{12} - x_1^{16}\}$. From Lemma 5.1, since x_1 divides x_1x_2 which is the leading monomial of the element $x_1x_2 - y_1^3 \in G'$, the* *monomial curve C obtained by a gluing of C_1 and C_2 does not have a Cohen-Macaulay tangent cone.*

This example leads us to ask the following question:

> *Question.* If two monomial curves have Cohen-Macaulay tangent cones, under which conditions does the monomial curve obtained by gluing these two monomial curves also have a Cohen-Macaulay tangent cone?

To answer this question partly, we first give the definition of a nice gluing, which generalizes the definition of a nice extension given in [4].

Definition 5.5 *Let $S_1 =< m_1, \dots, m_l >$ and $S_2 =< n_1, \dots, n_k >$ be two numerical semigroups minimally generated by $m_1 < \cdots < m_l$ and $n_1 < \cdots < n_k$ respectively. The numerical semigroup $S =< qm_1, \dots, qm_l, pn_1, \dots, pn_k >$ obtained by gluing S_1 and S_2 is called a nice gluing, if $p = b_1 m_1 + \cdots + b_l m_l \in S_1$ and $q = a_1 n_1 \in S_2$ with $a_1 \leq b_1 + \cdots + b_l$.*

Remark 5.6 *Notice that a nice extension defined in [4] is exactly a nice gluing with $S_2 =< 1 >$.*

Remark 5.7 *It is important to determine the smallest integer among the generators of the numerical semigroup $S =< qm_1, \dots, qm_l, pn_1, \dots, pn_k >$ obtained by gluing, since this is essential in checking the Cohen-Macaulayness of the tangent cone of the associated monomial curve. The condition $a_1 \leq b_1 + \cdots + b_l$ with $m_1 < \cdots < m_l$, $n_1 < \cdots < n_k$ and $\gcd(p, q) = 1$ implies that*

$$qm_1 = a_1 n_1 m_1 \leq (b_1 + \cdots + b_l) n_1 m_1 < pn_1 = (b_1 m_1 + \cdots + b_l m_l) n_1$$

and qm_1 is the smallest integer among the generators of S.

We are now ready to state the following:

Theorem 5.8 *Let $S_1 =< m_1, \ldots, m_l >$ and $S_2 =< n_1, \ldots, n_k >$ be two numerical semigroups minimally generated by $m_1 < \cdots < m_l$ and $n_1 < \cdots < n_k$, and let $S =< qm_1, \ldots, qm_l, pn_1, \ldots, pn_k >$ be a nice gluing of S_1 and S_2. If the associated monomial curves $C_1 = C(m_1, \ldots, m_l)$ and $C_2 = C(n_1, \ldots, n_k)$ have Cohen-Macaulay tangent cones at the origin, then the monomial curve $C = C(qm_1, \ldots, qm_l, pn_1, \ldots, pn_k)$ has also Cohen-Macaulay tangent cone at the origin, and thus, the Hilbert function of the local ring $K[[t^{qm_1}, \ldots, t^{qm_l}, t^{pn_1}, \ldots, t^{pn_k}]]$ is non-decreasing.*

Proof: By using the notation in [32], we denote the s-polynomial of the polynomials f and g by $\mathrm{spoly(f, g)}$ and the Mora's polynomial weak normal form of f with respect to G by $NF(f|G)$. Let $G_1 = \{f_1, \ldots, f_s\}$ be a minimal standard basis of the ideal $I(C_1) \subset K[x_1, \ldots, x_l]$ with respect to the negative degree reverse lexicographical ordering with $x_2 > \cdots > x_l > x_1$ and $G_2 = \{g_1, \ldots, g_t\}$ be a minimal standard basis of the ideal $I(C_2) \subset K[y_1, \ldots, y_k]$ with respect to the negative degree reverse lexicographical ordering with $y_2 > \cdots > y_k > y_1$. Since C_1 and C_2 have Cohen-Macaulay tangent cones at the origin, we conclude from Lemma 5.1 that x_1 does not divide the leading monomial of any element in G_1 and y_1 does not divide the leading monomial of any element in G_2 for the given orderings. The defining ideal of the monomial curve C obtained by gluing is generated by the set $G = \{f_1, \ldots, f_s, g_1, \ldots, g_t, x_1^{b_1} \ldots x_l^{b_l} - y_1^{a_1}\}$. Moreover, this set is a minimal standard basis with respect to the negative degree reverse lexicographical ordering with $y_2 > \cdots > y_k > y_1 > x_2 > \cdots > x_l > x_1$, because $NF(\mathrm{spoly}(f_i, g_j)|G) = 0$, $NF(\mathrm{spoly}(f_i, x_1^{b_1} \ldots x_l^{b_l} - y_1^{a_1})|G) = 0$ and $NF(\mathrm{spoly}(g_j, x_1^{b_1} \ldots x_l^{b_l} - y_1^{a_1})|G) = 0$ for $1 \leq i \leq s$ and $1 \leq j \leq t$. This is due to the fact that $NF(\mathrm{spoly}(f, g)|G) = 0$, if $\mathrm{lcm(LM(f), LM(g)) = LM(f) \cdot LM(g)}$. From Remark 5.7, qm_1 is the smallest integer among the generators of G. Thus, C has Cohen-Macaulay tangent cone at the origin if and only if x_1, which corresponds to qm_1, is not a zero-divisor in $K[x_1, \ldots, x_l, y_1, \ldots, y_k]/I(C)_*$. Since x_1 does not divide the leading monomial of any element in G_1 and G_2, and $\mathrm{LM}(x_1^{b_1} \ldots x_l^{b_l} - y_1^{a_1}) = y_1^{a_1}$, x_1 does not divide the leading monomial of any element in G, which is a minimal standard basis with respect to the negative degree reverse lexicographical ordering with $y_2 > \cdots > y_k > y_1 > x_2 > \cdots > x_l > x_1$. Thus, from Lemma 5.1, C has

Cohen-Macaulay tangent cone at the origin. $\square$

Remark 5.9 *From Remark 5.6, every nice extension is a nice gluing. Thus, if the monomial curve $C = C(m_1, \ldots, m_l)$ has a Cohen-Macaulay tangent cone at the origin, then every nice extension $C' = C(qm_1, \ldots, qm_l, b_1m_1 + \cdots + b_lm_l)$ of C has also Cohen-Macaulay tangent cone at the origin. Therefore, Theorem 5.8 generalizes the results in [2, Proposition 4.1] and [4, Theorem 3.6].*

Example 5.10 *Let C_1 and C_2 be the monomial curves $C_1 = C(m_1, m_2)$ with $m_1 < m_2$ and $C_2 = C(n_1, n_2)$ with $n_1 < n_2$. Obviously, they have Cohen-Macaulay tangent cones. From Theorem 5.8, every monomial curve $C = C(qm_1, qm_2, pn_1, pn_2)$ obtained by a nice gluing with $q = a_1n_1$, $p = b_1m_1 + b_2m_2$, $\gcd(p, q) = 1$ and $a_1 \leq b_1 + b_2$ has Cohen-Macaulay tangent cone at the origin, so the local ring $R = K[[t^{qm_1}, t^{qm_2}, t^{pn_1}, t^{pn_2}]]$ associated to the monomial curve C has a non-decreasing Hilbert function. Thus, by starting with fixed m_1, m_2, n_1 and n_2, we can construct infinitely many families of 1-dimensional local rings with non-decreasing Hilbert functions. For example, consider the monomial curves $C_1 = C(2, 3)$ and $C_2 = C(4, 5)$. By choosing $q = 2n_1 = 8$ and $p = (2r)m_1 + m_2 = 4r + 3$, for any $r \geq 1$, we obtain the monomial curve $C(16, 24, 16r + 12, 20r + 15)$, which is a nice gluing of C_1 and C_2. Since C is also a complete intersection monomial curve having a Cohen-Macaulay tangent cone, the associated local rings are Gorenstein with non-decrasing Hilbert functions, and that supports Rossi's conjecture saying that a one-dimensional Gorenstein local ring has a non-decreasing Hilbert function [4].*

This example shows that gluing is an effective method to obtain new families of monomial curves with Cohen-Macaulay tangent cones. Especially in affine 4-space, nice gluing is a very efficent method to obtain large families of complete intersection monomial curves with Cohen-Macaulay tangent cones, since every monomial curve in affine 2-space has a Cohen-Macaulay tangent cone.

5.3 A Conjecture

It is also possible to construct large families of gluings, which are not nice, but still give families of monomial curves with associated local rings having non-decreasing Hilbert functions.

Example 5.11 *Let C_1 and C_2 be the monomial curves $C_1 = C(5,12)$ and $C_2 = C(7,8)$. Obviously, they have Cohen-Macaulay tangent cones and thus their associated local rings have non-decreasing Hilbert functions. The family of monomial curves*

$$C = C(5 \cdot 7 \cdot (2d+1), 12 \cdot 7 \cdot (2d+1), 7 \cdot 17 \cdot d, 8 \cdot 17 \cdot d)$$

for $d \geq 1$ and d not divisible by 7 is a gluing, but not a nice gluing, of C_1 and C_2. Computations with Singular [31] show that, for $1 \leq d \leq 4$, C does not have a Cohen-Macaulay tangent cone, but its associated local ring has a non-decreasing Hilbert function. (Note that $d = 1$ gives Example 5.2.) For $d \geq 5$ and d not divisible by 7, the generator set $G = \{x_1^{12} - x_2^5, y_1^8 - y_2^7, y_1^{2d+1} - x_1^{12+d}x_2^{d-5}\}$ of $I(C)$ is a minimal standard basis with respect to the negative degree reverse lexicographical ordering with $x_1 < y_2 < y_1 < x_2$. Since x_1 does not divide the set $\{x_2^5, y_2^7, y_1^{2d+1}\}$ of leading monomials of the polynomials in the set G, C has Cohen-Macaulay tangent cone at the origin by Lemma 5.1. As a result, the Hilbert function of the local ring $R = K[[t^{5\cdot 7\cdot(2d+1)}, t^{12\cdot 7\cdot(2d+1)}, t^{7\cdot 17\cdot d}, t^{8\cdot 17\cdot d}]]$ associated to the monomial curve C is non-decreasing for $d \geq 1$ and d not divisible by 7. Again notice that for each d, C is a complete intersection monomial curve, and this result also supports Rossi's conjecture.

All these results and computations give examples of local rings, which have non-decreasing Hilbert functions and which are associated to monomial curves obtained by a gluing or a nice gluing of two monomial curves with associated local rings having non-decreasing Hilbert functions. Thus, depending on this idea, we formulate the following conjecture:

Conjecture 5.12 *If the Hilbert functions of the local rings associated to two* *complete intersection monomial curves are non-decreasing, then the Hilbert function of the local ring associated to the monomial curve obtained by gluing these two monomial curves is also non-decreasing.*

We know that every monomial curve in affine 2-space is obtained by gluing two complete intersection monomial curves $C_1 = C(1)$ and $C_2 = C(1)$ both having Cohen-Macaulay tangent cones obviously, and it is easy to check that every local ring associated to a monomial curve in affine 2-space has a non-decreasing Hilbert function. In affine 3-space, every monomial curve is not obtained by gluing, but every local ring associated to a monomial curve in affine 3-space has also a non-decreasing Hilbert function. This follows from the important result of Elias saying that every one-dimensional Cohen-Macaulay local ring with embedding dimension three has a non-decreasing Hilbert function [21]. Thus, the answer to the above conjecture is positive for the monomial curves in affine 2-space and 3-space, which are obtained by gluing, while the conjecture is open even for the complete intersection monomial curves in 4-space, which are obtained by gluing. What makes this question important is that, if the answer is affirmative, it will have been proved that the Hilbert function of every local ring associated to any complete intersection monomial curve is non-decreasing. This will be due to a result of Delorme [17], which is restated by Rosales in terms of gluing and says that every complete intersection numerical semigroup minimally generated by at least two elements is a gluing of two complete intersection numerical semigroups [65, Theorem 2.3]. Considering that it is still not known whether the Hilbert function of local rings with embedding dimension four associated to complete intersection monomial curves in affine 4-space is non-decreasing, this will be an important step in proving the conjecture due to Rossi saying that a one-dimensional Gorenstein local ring has a non-decreasing Hilbert function.

5.4 Hilbert functions via Free Resolutions

In this section we give an approach to study Hilbert functions using free resolutions. The advantage of this is that one can still get non-decreasing Hilbert function in the case where the tangent cone is NOT Cohen-Macaulay.

Let R be an $\mathbb{N}$-graded ring. Given a finitely generated $\mathbb{N}$-graded module M over R, the Hilbert function of M is defined to be $H_M(r) = dim_K(M_r)$, for all $r \in \mathbb{N}$.

If M has a minimal finite graded free resolution

$$0 \to F_d \to \cdots \to F_1 \to F_0 \to M \to 0$$

then the Hilbert Function of M is given by $H_M(r) = \sum_{i=0}^{d} H_{F_i}(r)$, where the free modules $F_i = \bigoplus_{j\in\mathbb{N}} R(-j)^{\beta_{i,j}}$, for all $i = 0, \ldots, d = projdim(M)$. Moreover the Hilbert series of M is given by $[\sum_{r\geq 0} \binom{n-1+r}{n-1} t^r][\sum_{i,j}(-1)^j \beta_{i,j} t^j]$.

One can use this approach to show that nice extensions of monomial curves with non-decreasing Hilbert functions have non-decreasing Hilbert function as well. For instance, if $C = C(6q, 7q, 15q, m)$ is a nice extension of $C(6, 7, 15)$, that is $m = 6b_1 + 7b_2 + 15b_3$ and $q \leq b_1 + b_2 + b_3$, then we show that its Hilbert function is non-decreasing. Note that tangent cones of these monomial curves are NOT Cohen-Macaulay.

Hilbert functions of certain extensions can be computed using a computer program such as Macaulay and Singular. For instance, the following sequence of numbers describe the Hilbert function of extensions where $1 < q < 7$:

$$1, 4, 7, 9, 10, 11, 12, 12, 12, 12, 12, \ldots$$

$$1, 4, 8, 12, 14, 16, 17, 18, 18, 18, 18, \ldots$$

$$1, 4, 8, 13, 17, 20, 22, 23, 24, 24, 24, \ldots$$

$$1, 4, 8, 13, 18, 23, 26, 28, 29, 30, 30, \ldots$$

$$1, 4, 8, 13, 18, 24, 29, 32, 34, 35, 36, \ldots$$

Obviously, they are non-decreasing. For the extensions where $q \geq 7$, we use free resolutions of their tangent cones.

Let $R = K[x_1, x_2, x_3, x_4]$ and $M = R/I(C)^*$. A minimal free resolution of M is as follows

$$0 \to F_4 \to F_3 \to F_2 \to F_1 \to F_0 \to M \to 0$$

where $F_0 = R$, $F_1 = R(-2)^2 \bigoplus R(-4) \bigoplus R(-6) \bigoplus R(-q)$,
$F_2 = R(-3) \bigoplus R(-5)^2 \bigoplus R(-7) \bigoplus R(-q-2))^2 \bigoplus R(-q-4) \bigoplus R(-q-6)$,
$F_3 = R(-6) \bigoplus R(-q-3) \bigoplus R(-q-5)^2 \bigoplus R(-q-7)$ and $F_4 = R(-q-6)$.

In the sequel, if $a < b$ then we assume that $\binom{a}{b} = 0$. Thus, we have

$$\begin{aligned}
H_M(r) &= H_{F_0}(r) - H_{F_1}(r) + H_{F_2}(r) - H_{F_3}(r) + H_{F_4}(r) = \\
&= \binom{r+3}{3} - [2\binom{r+1}{3} + \binom{r-1}{3} + \binom{r-3}{3} + \binom{r-q+3}{3}] + \\
&+ [\binom{r}{3} + 2\binom{r-2}{3} + \binom{r-4}{3} + 2\binom{r-q+1}{3} + \binom{r-q-1}{3} + \\
&+ \binom{r-q-3}{3}] - [\binom{r-3}{3} + \binom{r-q}{3} + 2\binom{r-q-2}{3} + \\
&+ \binom{r-q-4}{3}] + \binom{r-q-3}{3} = \\
&= \binom{r+3}{3} - 2\binom{r+1}{3} + \binom{r}{3} - \binom{r-1}{3} + 2\binom{r-2}{3} - 2\binom{r-3}{3} + \\
&+ \binom{r-4}{3} - \binom{r-q+3}{3} + 2\binom{r-q+1}{3} - \binom{r-q}{3} + \\
&+ \binom{r-q-1}{3} - 2\binom{r-q-2}{3} + 2\binom{r-q-3}{3} - \binom{r-q-4}{3} = \\
&= [\binom{r+3}{3} - \binom{r+1}{3}] - [\binom{r+1}{3} - \binom{r}{3}] - [\binom{r-1}{3} - \binom{r-2}{3}] + \\
&+ [\binom{r-2}{3} - \binom{r-3}{3}] - [\binom{r-3}{3} - \binom{r-4}{3}] - [\binom{r-q+3}{3} - \\
&- \binom{r-q+1}{3}] + [\binom{r-q+1}{3} - \binom{r-q}{3}] + [\binom{r-q-1}{3} - \binom{r-q-2}{3}] - \\
&- [\binom{r-q-2}{3} - \binom{r-q-3}{3}] + [\binom{r-q-3}{3} - \binom{r-q-4}{3}].
\end{aligned}$$

If $r < q$ then all the combinations above involving q are equal to zero. As a result Hilbert function becomes

$$\begin{aligned}
H_M(r) &= [\binom{r+3}{3} - \binom{r+1}{3}] - [\binom{r+1}{3} - \binom{r}{3}] - [\binom{r-1}{3} - \binom{r-2}{3}] + \\
&+ [\binom{r-2}{3} - \binom{r-3}{3}] - [\binom{r-3}{3} - \binom{r-4}{3}].
\end{aligned}$$

Using $\binom{a+1}{b} - \binom{a}{b} = \binom{a}{b-1}$, $\binom{a+2}{b} - \binom{a}{b} = 2\binom{a}{b-1} + \binom{a}{b-2}$ and $\binom{a+3}{2} - \binom{a}{2} = 3(a+1)$

we get

$$\begin{aligned}
H_M(r) &= [2\binom{r+1}{2} + \binom{r+1}{1}] - \binom{r}{2} - \binom{r-2}{2} + \binom{r-3}{2} - \binom{r-4}{2} \\
&= r+1+[\binom{r+1}{2} - \binom{r}{2}] + [\binom{r+1}{2} - \binom{r-2}{2}] + [\binom{r-3}{2} - \binom{r-4}{2}] \\
&= [r+1+r+3r-3+r-4] \\
&= [6r-6], \ \text{ for all } \ r < q, \text{which is non-decreasing.}
\end{aligned}$$

When $r \geq q+4$ we similarly find that

$$\begin{aligned}
H_M(r) &= [2\binom{r+1}{2} + \binom{r+1}{1}] - \binom{r}{2} - \binom{r-2}{2} + \binom{r-3}{2} - \binom{r-4}{2} - \\
&- [2\binom{r-q+1}{2} + \binom{r-q+1}{1}] + \binom{r-q}{2} + \binom{r-q-2}{2} - \\
&- \binom{r-q-3}{2} + \binom{r-q-4}{2} = \\
&= r+1+[\binom{r+1}{2} - \binom{r}{2}] + [\binom{r+1}{2} - \binom{r-2}{2}] + \\
&+ [\binom{r-3}{2} - \binom{r-4}{2}] - (r-q+1) - [\binom{r-q+1}{2} - \binom{r-q}{2}] - \\
&- [\binom{r-q+1}{2} - \binom{r-q-2}{2}] - [\binom{r-q-3}{2} - \binom{r-q-4}{2}] = \\
&= [6r-6] - [6r-6q-6] = 6q, \ \text{ for all } \ r \geq q+4,
\end{aligned}$$

which is non-decreasing as well. One can compute the following values directly using the formula of Hilbert function above:

$H_M(q) = 6q-7$, $H_M(q+1) = 6q-4$, $H_M(q+2) = 6q-2$, and $H_M(q+3) = 6q-1$. Hence, Hilbert functions of all nice extensions $C = C(6q, 7q, 15q, m)$, for all $q \geq 7$, are non-decreasing.

Chapter 6

Conclusion

We studied certain properties of monomial curves in this thesis. Namely, we investigate if they are set theoretic complete intersection and if their Hilbert function is non-decreasing. We introduce and discuss certain properties of extensions of monomial curves. We have seen that the algebraic structure of affine extensions are easy to determine contrary to the case of projective extensions. That is why we have used affine parts of the projective extensions to conclude that they are set theoretic complete intersections, a geometric property. This is also valid for projection of toric ideals, that is, the relation between the toric ideals in question is mysterious in general. Our experiences with gluing technique suggest that knowing the algebraic description of the ideal helps to understand the geometry of the toric variety. Therefore, a logical continuation may be to find the exact relation between a toric ideal and its projections. More precisely, it would be interesting to extend a minimal basis of a toric ideal to a minimal basis of its projection.

We have stated a conjecture saying that a gluing of two monomial curves whose Hilbert functions are non-decreasing has a non-decreasing Hilbert function. And we have shown particularly that the conjecture is true for nice extensions, a special type of gluing. Hence, another very natural continuation is to prove the conjecture which will imply that Hilbert functions of complete intersection monomial curves are non-decreasing.

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