INDECOMPOSABLE CYCLES ON A PRODUCT OF CURVES

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DOCTOR OF PHILOSOPHY

By

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May, 2012

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

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ABSTRACT

INDECOMPOSABLE CYCLES ON A PRODUCT OF CURVES

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In his pioneering work [3], S.Bloch introduced higher Chow groups, denoted by $CH^n(X, m)$ as a natural generalization of the classical case and generalized the Grothendieck amended Riemann Roch theorem to these groups, which states that the higher Chow ring and higher K-theory of a projective algebraic manifold are isomorphic working over rationals. This brilliant invention of Bloch brought a new insight to the study of algebraic cycles and K-theory. In this thesis, we study "interesting cycle classes" , namely indecomposable cycles for products of curves. In the case $m = 1$, indecomposable cycles are cycles in $CH^n(X, 1)$ which do not come from the image of the intersection pairing $CH^1(X, 1) \otimes CH^{n-1}(X)$. We prove that the group of indecomposable cycles, $CH^2_{ind}(X, 1; \mathbb{Q})$, is nontrivial for a sufficiently general product of two elliptic curves.

Keywords: Algebraic cycles, Rational equivalence, Chow group, Hodge conjecture, Cycle class maps, Higher Chow groups, Deligne cohomology, Regulators, Hodge-$\mathcal{D}$ conjecture, Indecomposable cycles.

ÖZET

EĞRİLERİN BİR ÇARPIMI ÜZERİNDE İNDİRGENEMEZ DÖNGÜLER

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Öncü çalışması [3] de, S. Bloch, $CH^n(X, m)$ ile gösterilen yüksek Chow döngülerini klasik durumun doğal bir genellemesi olarak tanımladı ve cebirsel izdüşümsel bir manifoldun rasyonel katsayılı yüksek Chow halkası ile yüksek K teorisinin eşyapısal olduğunu ifade eden, Grothendieck Riemann Roch teoremini genelledi. Bloch'un bu parlak buluşu cebirsel döngüler ve K teorisi çalışmalarına yeni bir bakış açısı getirdi. Bu tezde eğrilerin çarpımları için indirgenemez döngüler adı verilen "önemli döngü sınıflarını" çalıştık. İndirgenemez döngüler, yüksek Chow grubu $CH^n(X, m)$ içinde $CH^1(X, 1) \otimes CH^{n-1}(X)$ kesişim eşlemesinin görüntüsünden gelmeyen döngülerdir. Yeterince genel iki eliptik eğrinin çarpımı için indirgenemez döngüler grubu $CH^2_{ind}(X, 1; \mathbb{Q})$' nin boş olmadığını ispatladık.

Anahtar sözcükler: Cebirsel döngü, Rasyonel denklik, Chow grubu, Hodge Sanısı, Döngü sınıf gönderimleri, Yüksek Chow grubu, Deligne kohomolojisi, Düzenleyiciler, Hodge-$\mathcal{D}$ sanısı, İndirgenemez döngüler.

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Contents

Chapter 1

Introduction

1.0.1 Introduction(for the lay person)

The study of algebraic cycles, is not only a very fundamental and prominent subject in algebraic geometry but also has connections with different areas of mathematics.

The main subject of study is subvarieties of compact complex manifolds, under suitable equivalence relations. Algebraic varieties are common solution sets of a finite set of polynomials over any field in general and in particular over complex numbers. Polynomials arising from algebra, algebraic varieties owe their fundamental loyalty to algebra. On the other hand such an object can represent an elliptic curve or a complex Riemann surface, or more generally a projective algebraic manifold, and hence it is a geometric object in its own right. The techniques for studying such objects also heavily borrow from different areas of mathematics such as complex analysis, Hodge theory, number theory, topology.

One particular example of the interplay of geometry, algebra and analysis is the case of a compact Riemann surface M. Geometrically, M is a projective algebraic variety, Analytically it is a compact Riemann surface, and from the point of view of algebra, it has a meromorphic function field whose valuations are enough to reconstruct M as a geometric object.

To any such (complex) algebraic variety X, one associates algebraic datum that is intended to allow one to study X algebraically. In algebraic topology, this is typically singular (co)homology. Although that works well for topological spaces, singular (co)homology is not sensitive enough for algebraic varieties. One considers in this case an algebraic homology theory called Chow cohomology $CH^r(X)$, $r \geq 0$, which for suitable (nonsingular) X is a ring built out of subvarieties of X and intersection theory.

The free group generated by codimension k subvarieties of a projective algebraic manifold X, is called the group of codimension k algebraic cycles and denoted with $z^k(X)$. This group is too big to deal with, so one introduces equivalence relations on algebraic cycles. We will be interested in rational equivalence. Within the group of algebraic k cycles, there exists a special subgroup, the group of algebraic k cycles rationally equivalent to zero, denoted by $z^k_{rat}(X)$. The Chow group is the quotient $CH^k(X) := z^k(X)/z^k_{rat}(X)$.

The construction above is a natural generalization of divisors on Riemann surfaces and linear equivalence of divisors. A divisor on a Riemann surface X, is an element of the group generated freely by points in X. This group is actually the group zero 0-cycles on X; $z^1(X)$, so any divisor γ can be represented as $\gamma = \sum n_i p_i$ where $p_i \in X$ is a point and $n_i \in \mathbb{Z}$. Principal divisors, which are divisors of rational functions on X, form the subgroup of zero cycles rationally equivalent to zero, $z^1_{rat}(X)$ and the first Chow group of X is the group of divisors modulo principal divisors, i.e $CH^1(X) := z^1(X)/z^1_{rat}(X)$.

It turns out however that $CH^k(X)$ is hard to compute, and therefore one has to look at "realization maps" from $CH^k(X)$ to more computable homology theories. Such realizations are called regulators. For example the first Chow group $CH^1(X)$ can be identified with the Picard group of $Pic(X)$, which is the group of isomorphism classes of holomorphic line bundles on X, through cycle class map ϕ_1 (i.e; $\phi_1 : CH^1(X) \to Pic(X)$ is an isomorphism). In general cycle class map is far from being an isomorphism.

There is an abundance of eamples of regulators stemming from earlier literature in this subject. On the number theory side, there are the Dirichlet and more generally Borel regulators, which make use of the relation of algebraic K-theory to the subject of algebraic cycles. From this perspective, a regulator is often seen as a generalization of the classical logarithm. From a geometric point of view, one of the first examples of a regulator is the classical elliptic integral

$$p \in X \mapsto \int_{p_0}^{p} \frac{dx}{y} \in \mathbb{C}/\mathbb{Z}^2 \simeq X$$

where X is the (compactification of the) zeros of $y^2 = x^3 + bx + c$ (an elliptic curve). This is a multivalued integral, when viewed as a "map " to $\mathbb{C}$. Generalizations of this to all compact Riemann surfaces led to a crowning achievement in the late 19th century by Abel and Jacobi on their proof of Jacobi inversion. This led to a complete understanding of these multivalued integrals and their inverses. In the late 1960's Phillip Griffiths generalized this construction to a map

$$\phi_r : CH^r_{hom}(X) \to J^r(X),$$

where $CH^r_{hom}X \subset CH^r(X)$ is the subgroup of nullhomologous cycles, and $J^r(X)$ is a certain compact complex torus. A nullhomologous cycle is a cycle that yields no information from a singular (co)homology point of view. It is a generalization of this map that forms the central part of this thesis.

Another major development in the 1960's was A. Grothendieck's invention of algebraic K-theory. Like the Chow ring, this is a complicated object, denoted by $K_0(X)$, which have some appealing universal properties (related to the subject of motives). K-theory is related geometrically to X in terms of vector bundles over X, and has a natural λ-operation on it, for which one has an isomorphism induced by λ

$$K_0(X) \otimes \mathbb{Q} \xrightarrow{\sim} CH^\bullet(X) \otimes \mathbb{Q}$$

The map is called the Chern character map, and the corresponding isomorphism is called the Grothendieck-Riemann-Roch theorem. Grothendieck's invention would eventually have a far reaching generalization to the higher K-groups ($K_m(X)$) invented by D. Quillen. These higher K-groups still acquire a λ-operation, for which a cycle theoretic analogue was missing. It was Spencer Bloch's brilliant invention of his higher Chow groups $CH^r(X, m)$ in the 1980's which filled in the gap, and which led to Bloch's version of the Riemann-Roch theorem:

$$K_m(X) \otimes \mathbb{Q} \xrightarrow{\sim} CH^\bullet(X, m) \otimes \mathbb{Q}$$

Bloch (via his $CH^r(X, m)$) and Beilinson (via $K^m(X)$) independently constructed regulator maps

$$CH^r(X, m) \to H^{2r-m}(X, r),$$

where $H^{2r-m}(X, r)$ is any "reasonable" cohomology theory. The one major problem is that these maps are very hard to compute.

S. Bloch's Riemann-Roch theorem not only connects K-theory to the subject of algebraic cycles, but enables one to put the Griffiths Abel-Jacobi map, the Borel and Dirichlet maps under one umbrella, the aforementioned regulators.

In this thesis we will study the cycle class map between Bloch's higher Chow groups and appropriate Deligne cohomology groups;

$$c_{k,1} := CH^k(X, 1 : \mathbb{Q}) \to H^{2k-1}_{\mathcal{D}}(X, \mathbb{Q}(k))$$

and the "real" regulator;

$$r_{k,1} := CH^k(X, 1) \otimes \mathbb{Q} \to H^{2k-1}_{\mathcal{D}}(X, \mathbb{R}(k))$$

where X is a sufficiently general product of elliptic curves.

Elliptic curves and their products carry a rich geometry making them favorable objects in the study of indecomposable higher Chow cycles. Works of Lewis ([17], [6]), Muller-Stach([30]), Spiess ([35]), Mildenhal([28]) are some examples.

We construct interesting cycle classes called indecomposable cycles in higher Chow groups of a sufficiently general product of two elliptic curves. This result together with a theorem of Rosenschon and Saito ([33]) implies that the group of indecomposable cycles $CH^3(E_1 \times E_2 \times E_3, 1)_{ind}$ is uncountably generated fora sufficiently general product of three elliptic curves. Another corollary to our result is that the transcendental regulator is nontrivial for sufficiently general product of two elliptic curves.

1.0.2 Precise Results and Organization of the Thesis

The main subject of this thesis is to study higher algebraic cycles on certain algebraic varieties. Before studying the higher case, we take into consideration algebraic cycles in the classical sense. The first chapter is devoted to the explanation of basics of algebraic cycles, cycles class maps, classical Chow groups and the Hodge conjecture.

In the second chapter, we will define higher Chow groups, Deligne cohomology, and link them through higher cycle class maps and the real regulator. We will present the subject by emphasizing how it developed from the classical case.

Our results appear in the third chapter. The results obtained in this thesis, originate from a research project carried out with Prof. James Lewis from University of Alberta, focused on proving the results in [6]. After its publication, with a remark of Prof. M.Saito, it was understood that there is a crucial miscalculation which led to a fundamental error in [17]. Later some of the results stated in this paper were proved with different techniques ([6]) which supported the idea that the results and the approach in ([17]) works but needs some alterations.

Our main result is

Theorem 1.0.1. *$CH^2_{ind}(E_1 \times E_2, 1)$ is non trivial for a sufficiently general product $E_1 \times E_2$ of elliptic curves E_1 and E_2.*

The proof is based on a construction which uses the torsion points on elliptic curves and their properties.

The corollary below, which is also one of the most important results of [17], follows from our result and a theorem of Rosenschon and Saito ([33][Theorem 0.2])

Corollary 1.0.2. *Let X be a sufficiently general product of three elliptic curves, then $CH^3_{ind}(X, 1)$ is uncountable.*

Our last result is a corollary of our main result and Corollary 1.6 of [7]

Corollary 1.0.3. *Let X be a sufficiently general product of two elliptic curves, then the transcendental regulator $\phi_{2,1}$ is nontrivial.*

In the last chapter, first we study the error in [17]. In the second section of this chapter, we prove our main result. The last section is devoted to corollaries and consequences of our main result, and further research.

Chapter 2

Algebraic Cycles (Classical Scenario)

2.1 Preliminaries

Let $\mathbb{P}^n = \{\mathbb{C}^{n+1}/\{0\}\}/\mathbb{C}^\star$ be the complex projective n space. A projective algebraic manifold X is a closed embedded submanifold of $\mathbb{P}^n$. By a theorem of Chow, X is a smooth algebraic variety; X is the common zero locus of finitely many polynomials and the tangent space of X at all points has the same rank. Smoothness can also be expressed as the non-vanishing of the determinant of the Hessian matrix of second derivatives of polynomials defining X. Being projective X can be embedded in projective space $\mathbb{P}^n$ and inherits plenty of subvarieties lying in $\mathbb{P}^n$. Algebraic cycles are introduced to understand projective complex manifolds by studying their subvarieties and their geometry by means of intersection theory.

Definition 2.1.1. *A codimension r algebraic cycle V on X is a $\mathbb{Z}$ formal sum of codimension r irreducible subvarieties in X. Such a cycle can be written as $\sum_{codim_X V_i = r} n_i V_i$(where $n_i = 0$ except for finitely many V_i)*

The free Abeilan froup generated by codimension r irreducible subvarieties of X is denoted by $z^r(X)$. One can consider the dimension instead of codimension.

The free Abelian group generated by algebraic cycles of dimension n is denoted by $z_n(X)$. Notice that $z^r(X) = z_{n-r}(X)$. One has to carry out extra indice n, working with dimension notation so throughout this thesis we will use the codimension notation.

Example 2.1.2. $z_0(X) = z^n(X) = \{\sum n_i p_i; p_i \in X \textit{is a point}, n_i \in \mathbb{Z}\}$
$z_1(X) = z^{n-1}(X) = \{\sum n_i C_i; C_i \subseteq X \textit{is a curve}, n_i \in \mathbb{Z}\}$

Algebraic cycles are formal sums of subvarieties, (like points and curves as in the example), so we can integrate suitable differential forms on them. Next we will discuss briefly the differential data associated to a complex projective manifold X.

Let E_X^k be the vector space of $\mathbb{C}$-valued $\mathbb{C}^\infty$ k-forms on X. Any complex valued k form can be decomposed into holomorphic and antiholomorphic parts. In local coordinates $z = (z_1, \cdots, z_n)$ on X, a complex k-form ω can be expressed as

$$\begin{aligned} \omega = \sum_{|I|=p,|J|=q} f_{IJ} dz_I \wedge d\overline{z}_J \text{ where} \\ |I| = \{1 \leq i_1 < \cdots i_p \leq n\} \\ |J| = \{1 \leq j_1 < \cdots j_q \leq n\}, \\ dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p} \\ d\overline{z}_J = d\overline{z}_{j_1} \wedge \cdots \wedge \quad d\overline{z}_{j_q}. \end{aligned}$$

The decomposition of complex forms into holomorphic and antiholomorphic parts also carries out to the vector spaces level;

$$E_X^k = \bigoplus_{p+q=k} E_X^{p,q}$$

where $E_X^{p,q}$ is the vector space of $\mathbb{C}^\infty$ (p,q) forms, having p holomorphic and q antiholomorphic differentials. Moreover if ω is a (p,q) form then its complex conjugate $\overline{\omega}$ is a (q,p) form, so we have an isomorphism of the vector spaces

$$\overline{E_X^{p,q}} \simeq E_X^{q,p}.$$

There exists the total, holomorphic and antiholomorphic differentiation operators;

$$d : E_X^k \to E_X^{k+1}$$

$$\partial : E_X^{p,q} \to E_X^{p+1,q}$$

$$\overline{\partial} : E_X^{p,q} \to E_X^{p,q+1}$$

acting on these vector spaces. These operators are boundary operators; $d^2 = \partial^2 = \overline{\partial}^2 = 0$ and satisfy the relations $d = \partial + \overline{\partial}$ and $\partial\overline{\partial} + \overline{\partial}\partial = 0$. The cohomology of the resulting complexes yield de Rham and Dolbeault cohomologies

Definition 2.1.3. *De Rham cohomology is defined to be the homology of the complex $(E_X^\bullet, d)$;*

$$H_{dr}^k(X, \mathbb{C}) = \frac{kerd : E_X^k \to E_X^{k+1}}{Imd : E_X^{k-1} \to E_X^k}$$

Moreover since we are dealing with projective manifolds X, the boundary operators ∂ and $\overline{\partial}$ respect the decomposition of complex forms in to (p, q) forms. Dolbeault or (p, q)-cohomology can be defined as

$$H^{p,q}(X, \mathbb{C}) = \frac{\{\omega \in E_X^{p,q}; d\omega = 0\}}{Im\partial\overline{\partial} : E_X^{p-1,q-1} \to E_X^{p,q}}.$$

The de Rham theorem we state below, establishes the link between the differential/analytic data encoded in de Rham cohomology groups and the topological data encoded in singular cohomology.

Theorem 2.1.4 (De Rham [18](p. 43))**.**

$$H_{sing}^k(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \simeq H_{dr}^k(X, \mathbb{C})$$

We will drop the subscript dr to denote de Rham cohomology, unless the distinction is necessary. The decomposition of complex forms and the symmetry property of underlying vector spaces also carries out to the cohomology level, and De Rham cohomology groups split into a sum of Dolbeoult cohomology groups.

Theorem 2.1.5 (Hodge Decomposition Theorem [18](p. 116))**.**

$$H_{dr}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X, \mathbb{C}) \qquad \overline{H^{p,q}(X, \mathbb{C})} \simeq H^{q,p}(X, \mathbb{C})$$

An immediate consequence of Hodge decomposition theorem is that odd degree cohomology groups of a projective algebraic manifold have even dimension. For $n = 2k+1$,

$$H^{2k+1}(X,\mathbb{C}) = H^{0,2k+1}(X,\mathbb{C}) \oplus H^{1,2k}(X,\mathbb{C}) \cdots \oplus H^{k,k+1}(X,\mathbb{C}) \oplus \\ H^{k+1,k}(X,\mathbb{C}) \oplus \cdots H^{2k,1}(X,\mathbb{C}) \oplus H^{2k+1,0}(X,\mathbb{C})$$

There are $2k+2$ terms in the decomposition and the complex conjugate of each vector space in the first half appears in the second half.

A $2n$ differential form can be integrated on an n dimensional complex projective algebraic manifold. This provides a non-degenerate pairing between complementary dimensional de Rham and Dolbeault cohomologies. The pairings;

$$H^k_{dr}(X,\mathbb{C}) \otimes H^{2n-k}_{dr}(X,\mathbb{C}) \to \mathbb{C} \text{ and}$$

$$H^{p,q}_{dr}(X,\mathbb{C}) \otimes H^{n-p,n-q}_{dr}(X,\mathbb{C}) \to \mathbb{C}$$

induced by

$$(\omega_1,\omega_2) \to \int_X \omega_1 \wedge \omega_2$$

are non-degenerate [18](p. 59). These non-degenerate pairings induce following isomorphisms between corresponding cohomology groups;

$$H^k_{dr}(X,\mathbb{C}) \simeq (H^{2n-k}_{dr}(X,\mathbb{C}))^\vee$$

$$H^{p,q}_{dr}(X,\mathbb{C}) \simeq (H^{n-p,n-q}_{dr}(X,\mathbb{C}))^\vee$$

which are known as Poincaré and Serre dualities.

2.2 The Cycle Class Map and the Hodge Conjecture

The link between the algebraic data encoded in the group of algebraic cycles and differential/analytic data encoded in de Rham cohomology groups are maps called cycle class maps. We will define the first cycle map, denoted by cl_k, first.

The first cycle class map, sends algebraic k cycles to $2k$ dimensional de Rham cohains.

$$cl_k : z^k(X) \to H^{2k}_{dr}(X, \mathbb{C}) \simeq (H^{2n-2k}_{dr}(X, \mathbb{C}))^{\vee}$$

A codimension k algebraic cycle can be represented as $\sum_i n_i V_i$ where n_i's are integers and V_i's are codimension k irreducible subvarieties in X. For simplicity, we will define the cycle class map for an irreducible codimension k subvariety V and then extend it by linearity. Let $\{\omega\}$ be a cohomology class in $H^{2n-2k}_{dr}(X, \mathbb{C})$, then the first cycle class map is defined via the relation

$$cl_k(V)\{\omega\} = \int_{V \setminus V_{sing}} \omega$$

In order to prove that this map is well-defined, we must prove that the result of the integral is finite, and it does not depend on the representative of the cohomology class.

Consider a desingularization of V, $f : V^* \to V$, such that $f^{-1}(V_{sing})$ is normal crossing divisor. Then the singular locus of V has measure zero; $codim_{V^*} f^{-1}(V_{sing}) \geq 1$. The desingularization V^* is compact, therefore the integral

$$\int_{V \setminus V_{sing}} \omega = \int_{V^*} f^*(\omega)$$

is finite.

Two different representatives of the same cohomology class $\{\omega\} \in (H^{2n-2k}_{dr}(X, \mathbb{C}))^{\vee}$ will differ by a closed form; $\omega_1 - \omega_2 = d\eta$. The cycle class image of a closed form $d\eta$ is;

$$\int_{V \setminus V_{sing}} d\eta = \int_{V^*} f^*(d\eta) = \int_{\partial V^*} \eta = 0.$$

Hence the first cycle class map does not depend on the representative of the cohomology class.

By the Hodge decomposition theorem, the de Rham cohomology groups decomposes into Dolbeault cohomology groups;

$$H^k_{dr}(X,\mathbb{C}) = \bigoplus_{p+q=k} [H^{p,q}(X,\mathbb{C}) \simeq H^{n-p,n-q}(X,\mathbb{C})].$$

A natural question to consider is the following: Which (p,q) cohomology classes are contained in the image of the cycle class map? The answer can be found by matching dimensions of vector spaces of differential forms and algebraic cycles.

Let $\{\omega\}$ be a cohomology class in $(H^{n-p,n-q}_{dr}(X,\mathbb{C}))$ such that $cl_k(V)\{\omega\} \neq 0$ for some k cycle V. For simplicity let V be irreducible. Since V has complex dimension $n-k$, it can support at most $n-k$ holomorphic and $n-k$ antiholomorphic forms. So $p \leq k$ and $q \leq k$, and $p+q = 2k$. The only solution to this system of inequalities is $p = q = k$. Therefor only the middle cohomology $H^{k,k}_{dr}(X,\mathbb{C})$ is hit by the image of the cycle class map; $cl_k(z^k(X)) \subset H^{k,k}_{dr}(X,\mathbb{C})$.

Interpreting the first cycle class map as compositon of fundemantal class map in integral homology and Poincare duality it is esasy to conclude that the image of the first cycle class map lies in the integral cohomology $H^{2k}_{dr}(X,\mathbb{Z})$.

These two observations on the image of the first cycle class map rises the following the question: How far is this map from being an isomorphism? Hodge's original version of the conjecture is stated as above;

Conjecture 2.2.1. *The cycle class map*

$$cl_k : z^k(X) \mapsto H^{2k}(X,\mathbb{Z}) \cap H^{k,k}_{dr}(X,\mathbb{C})$$

is surjective.

The term $H^{2k}(X,\mathbb{Z}) \cap H^{k,k}_{dr}(X,\mathbb{C})$ is called the group of Hodge cycles and is denoted by $Hg^{k,k}(X,\mathbb{Z})$. The conjecture says that Hodge classes are algebraic.

In this form, Hodge conjecture is known to be false. A first counterexample was constructed by Atiyah and Hirzebruch [1]. They constructed a nonanalytic torsion class in $H^{2k}(X,\mathbb{Z}) \cap H^{k,k}_{dr}(X,\mathbb{C})$. The Integral Hodge Conjecture modulo torsion is also proven to be false by Kollar [25]. He constructed a class in $H^{2k}(X,\mathbb{Z}) \cap H^{k,k}_{dr}(X,\mathbb{C})$ which is not algebraic but an integral multiple is algebraic. The natural amendment is considering rational cohomology classes which gives us the celebrated Hodge Conjecture;

Conjecture 2.2.2 (Hodge Conjecture). *The cycle class map*

$$cl_k : z^k(X) \otimes \mathbb{Q} \mapsto H^{2k}(X,\mathbb{Q}) \cap H^{k,k}_{dr}(X,\mathbb{C})$$

is surjective for all k.

Even the statement of the classical Hodge conjecture reveals its beauty and importance. The object on the left hand side, the group $z^k(X)$ of codimension k cycles on X, is constructed out of subvarieties of a projective algebraic manifold X, and encodes the algebraic data attached to X. On the right hand side $H^{2k}(X,\mathbb{Q})$ is the image of singular cohomology, which is a topological construction, in de Rham cohomology and carries topological data. $H^{k,k}_{dr}(X,\mathbb{C})$ is constructed out of differentials on X, and captures differential/analytic data. In some sense, Hodge conjecture relates algebraic, topological and differential/analytic data associated to a projective algebraic manifold X.

The Hodge conjecture has a natural generalization, stated in terms of Hodge structures and filtrations on cohomology groups. We will not use these notions later in our work, so we will not discuss them here, (for a detailed discussion of the Hodge Conjecture, see [27]), but we state this generalization of the Hodge conjecture.

Conjecture 2.2.3 (Grothendieck amended General Hodge Conjecture).

$$GHC(p,l,X) : F^p_a H^l(X,\mathbb{Q}) = F^p_h H^l(X,\mathbb{Q})$$

where $F^p_h H^l(X,\mathbb{Q})$ is the largest Hodge structure in $\{F^p H^l(X,\mathbb{Q}) \cap H^l(X,\mathbb{Q})\}$ and $F^p_a H^l(X,\mathbb{Q})$ is the Gysin images of $\sigma_\star : H^{l-2q}(\tilde{Y},\mathbb{Q}) \to H^l(X,\mathbb{Q})$ with Y having pure codimension in $q \geq p$ in X, and $\tilde{Y}$ is a desingularization of Y

F_h^p and F_a^p can be considered as ' rational ' and ' arithmetic filtrations ' on rational coefficient cohomology. Also note that $p = k$ and $l = 2k$ is the classical Hodge Conjecture we have stated earlier.

2.2.1 Cases Where Hodge Conjecture Holds

We list some of the cases where the classical Hodge conjecture is known to be true in this section. For a detailed discussion of a more complete list, one may refer to [34] and [27].

A first result on the Hodge Conjecture is the Lefschetz theorem on 1-1 classes [26], which predates the Hodge Conjecture. This theorem states that any element in $Hg^{1,1}(X, \mathbb{Z})$ is the cohomology class of a divisor on X, (i.e; cycle class map is surjective for $k = 1$, Hodge conjecture holds for k=1). The group of Hodge cycles $Hg^{k,k}(X, \mathbb{Z})$ is defined as $Hg^{k,k}(X) := i^{-1}(H^{k,k}(X, \mathbb{C}))$ where for $\mathbb{Z}(k) = (2\pi i)^k\mathbb{Z}$

$$i : H^{2k}(X, \mathbb{Z}(k)) \to H^{2k}(X, \mathbb{C})).$$

Lefschetz's proof uses normal functions. Unfortunately the method of normal functions can not be generalized, because Jacobi inversion fails in general [19]. A different proof employs sheaf cohomology and the exponential exact sequence.

Due to the Hard Lefschetz theorem, if Hodge Conjecture holds for $Hg^{p,p}(X, \mathbb{Q})$ then it holds for $Hg^{n-p,n-p}(X, \mathbb{Q})$.

These two results together imply that Hodge Conjecture holds for surfaces and threefolds.

For projective space all the cohomology is generated by the class of a hyperplane. For Grassmanians, the cohomology is generated by Schubert cycles. Similar to these examples, for quadrics and flag varieties all of the cohomology comes from algebraic cycles. For such varieties, the Hodge conjecture clearly holds.

For a smooth hypersurface $X \subset \mathbb{P}^{n+1}$ of degree d, using the weak Lefschetz theorem applied to its inclusion into projective space, together with hard Lefschetz theorem, it can be shown that Hodge Conjecture holds except the middle cohomology. For the middle cohomology $Hg^{n,n}(X,\mathbb{Q})$, there are various results under restrictions on the dimension and degree of the hypersurface.

When X is a uniruled or a unirational fourfold, or a Fano complete intersection of degree 4, Conte and Murre proved that Hodge Conjecture holds for $Hg^{2,2}(X,\mathbb{Q})$ [10], [11].

For some classes of Abelian varieties, the Hodge conjecture is verified. In many of these cases, the cohomology ring of Hodge cycles $Hg^{\star,\star}(X,\mathbb{Q})$ is generated by level one, by elements in $Hg^{1,1}(X,\mathbb{Q})$, and Hodge conjecture holds for $Hg^{1,1}(X,\mathbb{Q})$ by Lefschetz $(1,1)$ theorem. Examples of Abelian varieties for which Hodge conjecture holds are self product elliptic curves, 'sufficiently general' abelian varieties and simple abelian varieties of prime dimension.

2.2.2 The Abel Jacobi Map: The Second Cycle Class Map

The group of algebraic cycles is in general very large; even the set of elements mapped to zero by the first cycle class map is very big. While studying the group of algebraic cycles via the first cycle class map, it is important to consider the cycles mapped to zero. We call the cycles mapped to zero by first cycle class map, as cycles homologically equivalent to zero, or " cycles homologous to zero" for short. We denote the group of cycle homologically equivalent to zero by $z^k_{hom}(X)$;

$$z^k_{hom}(X) := ker(cl_k : z^k(X) \to Hg^{k,k}(X,\mathbb{Z})).$$

We will construct a map from this group to a certain complex torus, called the Griffiths Jacobian, following Griffith's prescription [20]. First we define the Hodge filtration on de Rham cohomology groups;

$$F^r H^k(X,\mathbb{C}) = \oplus_{p+q=k,p\geq r} H^{p,q}(X).$$

Notice that odd indexed cohomology groups are even dimensional;

$$H^{2k-1}(X,\mathbb{C}) = F^k H^{2k-1}(X,\mathbb{C}) \oplus \overline{F^k H^{2k-1}(X,\mathbb{C})}.$$

The Griffiths Jacobian is defined as;

$$J^k(X) := \frac{H^{2k-1}(X,\mathbb{C})}{F^k H^{2k-1}(X,\mathbb{C}) \oplus H^{2k-1}(X,\mathbb{Z})}.$$

By Serre duality;

$$J^k(X) \simeq \frac{F^{n-k+1}H^{2n-2k+1}(X,\mathbb{C})^\vee}{H_{2n-2k+1}(X,\mathbb{Z})}$$

where the denominator $H_{2n-2k+1}(X,\mathbb{Z})$ is called group of periods and identified with its image in $F^{n-k+1}H^{2n-2k+1}(X,\mathbb{Z})^\vee$;

$$H_{2n-2k+1}(X,\mathbb{C}) \mapsto F^{n-k+1}H^{2n-2k+1}(X,\mathbb{C})^\vee$$

$$\{\xi\} \mapsto \left(\{\omega\} \in F^{n-k+1}H^{2n-2k+1}(X,\mathbb{C}) \mapsto \int_\xi \omega\right).$$

Griffiths' generalization of the Abel-Jacobi map Φ_k is defined as follows: Let $\xi \in z^k_{hom}(X)$, then $cl_k(\xi)(\{\omega\}) = 0$ for all $\omega \in H^{2n-2k}(X,\mathbb{C})$, so ξ bounds a $2n-2k+1$ real dimensional chain ξ in X. Let $\{\omega\} \in F^{n-k+1}H^{2n-2k+1}(X,\mathbb{C})$, we define;

$$\Phi_k(\xi)(\{\omega\}) = \int_\xi \omega$$

modulo periods. Similar to the first cycle class map this map, one can show that Abel Jacobi is well-defined.

2.3 Classical Chow Groups

It is natural to consider certain equivalence relations when studying algebraic cycles. There are several reasons for that. The group of algebraic cycles is generated freely by all subvarieties of a fixed dimension, so this group is 'too big'. Even though the group $z^k(X)$ encodes the information about subvarieties, it does not reflect the geometry of X. The intersection properties, mutual positions of subvarieties is not captured by this group. To study this group, cycle class maps are introduced. The image of $z^k(X)$ lies in the cohomology ring of X, $H^\bullet(X) := \oplus H^k(X, \mathbb{C})$. Let $z^\bullet(X) := \oplus z^k(X)$, then the first cycle class maps cl_k for $k = 1, 2, \cdots, n$ maps $z^\bullet(X)$ to $H^\bullet(X)$. The collection of algebraic cycle groups $z^\bullet(X)$ does not have a multiplicative structure whereas the image $H^\bullet(X)$ is a ring and has a multiplication. Defining appropriate equivalence relations on algebraic cycles helps to deal with these problems. We will be interested in rational equivalence;

Definition 2.3.1. *Two algebraic cycles $\xi_1, \xi_2 \in z^k(X)$ are called rationally equivalent (denoted by $\xi_1 \sim_{rat} \xi_2$), if there exists a cycle $\omega \in z^k(\mathbb{P}^1 \times X)$ in " general position" such that $\omega(0) - \omega(\infty) = \xi_1 - \xi_2$. A cycle $\omega \in z^k(\mathbb{P}^1 \times X)$ is in general position if the cycle $Pr_{2,\star}(\omega.(t \times X)) \in z^k(X)$ is defined for each fiber.*

Notice that in the case of divisors, rational and linear equivalences coincide, so rational equivalence is a natural generalization of linear equivalence for divisors. If $\mathbb{P}^1$ is replaced with a smooth connected curve Γ in the definition above, and 0 and ∞ are replaced by any two points $P, Q \in \Gamma$ one gets the definition of algebraic equivalence. We consider the equivalence class of algebraic cycles with respect to these equivalence relations.

We denote the group of algebraic cycles rationally equivalent to zero as

$$z^k_{rat}(X) := \{\xi \in z^k(X) | \xi \sim_{rat} 0\}$$

and the group of algebraic cycles algebraically equivalent to zero as

$$z^k_{alg}(X) := \{\xi \in z^k(X) | \xi \sim_{alg} 0\}.$$

We also have homological equivalence introduced in the previous section. It is clear that;

$$z^k_{rat}(X) \subset z^k_{alg}(X) \subset z^k_{hom}(X).$$

This last inclusion also implies that $cl_k(z^k_{rat}(X)) = 0$.

We will be interested in rational equivalence, so we give another definition/characterization of it.

An algebraic cycle $\xi \in z^k(X)$ is rationally equivalent to zero if and only if ξ can be written as a sum of divisors of rational functions $f_i \in \mathbb{C}(Y_i)^\times$, where Y_i is a $k-1$ codimensional subvariety in X($\xi \sim_{rat} 0 \Leftrightarrow \xi = \sum_{i=1}^{n} div_{Y_i}(f_i)$ where $f_i \in \mathbb{C}(Y_i)^\times$ and $codim_X(Y_i) = k-1$) [21]

We define the Chow group as algebraic cycles modulo rational equivalence;

$$CH^k(X) := z^k(X)/z^k(X)_{rat},$$

and the Chow ring as the graded ring

$$CH^\bullet(X) := \bigoplus_{k=1}^{n} CH^k(X).$$

Similarly, we define the Chow group of cycles algebraically and homologically equivalent to zero as

$$CH^k_{alg}(X) := z^k_{alg}(X)/z^k(X)_{rat}; \qquad CH^k_{hom}(X) := z^k_{hom}(X)/z^k(X)_{rat}.$$

The product in the Chow ring comes from the intersection pairing. Moving elements in their equivalence class allows one to define a well-defined intersection pairing. For a detailed discussion of intersection theory see [15]. The Chow ring is a cohomology theory constructed out of subvarieties and their intersection properties for a projective algebraic manifold.

In general, it is quite complicated to compute the Chow ring of a projective algebraic manifold by studying its subvarieties. So the Chow ring is studied by the help of maps from the Chow groups to " more computable" cohomology theories.

An example of such a situation is obtained when cycle class maps are extended to Chow groups. We get the following commutative diagram;

$$\begin{array}{ccccccccc}
0 & \longrightarrow & CH^k_{hom}(X) & \longrightarrow & CH^kX & \longrightarrow & \frac{CH^k(X)}{CH^k_{hom}X} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \phi_k} & & \downarrow{\scriptstyle \varphi_k} & & \downarrow{\scriptstyle cl_k} & & \\
0 & \longrightarrow & J^k(X) & \longrightarrow & H^{2k}_{\mathcal{D}}(X, \mathbb{Z}(k)) & \longrightarrow & Hg^k(X) & \longrightarrow & 0
\end{array}$$

The group $H^{2k}_{\mathcal{D}}(X, \mathbb{Z}(k))$ is the Deligne cohomology and will be defined in the next chapter.

This diagram is commutative, and in the case $k = 1$, the map φ_1 is an isomorphism. In general the picture is quite complicated and φ_k is far from being an isomorphism.

One can define higher cycle class maps on $CH^k(X)$ each defined on the kernel of the previous ones into higher intermediate Jacobians, which leads to a filtration on classical Chow groups. Describing the Chow groups in terms filtrations, (as a realization of the conjectural motivic filtration) is a subject of central interest in this field.

Chapter 3

The Higher Case

In mid 50's Alexander Grothendieck introduced the K-groups which can be considered as the starting point of algebraic K-theory. He established the isomorphism between the Grothendieck group K_0 and the classical Chow ring, via the Chern character map, which is known as Grothendieck's version of the Riemann-Roch Theorem [5].

$$K_0(X) \otimes \mathbb{Q} \xrightarrow{\sim} CH^\bullet(X) \otimes \mathbb{Q}$$

Later, D. Quillen introduced the higher K-groups $K_m(X)$ [32].

On the other hand, in the 1980's Spencer Bloch invented the higher Chow groups and established the relation with Quillen's higher K-theory [3]. Bloch's work completed the whole picture. This result is called Bloch's version of the Riemann-Roch Theorem;

$$K_m(X) \otimes \mathbb{Q} \xrightarrow{\sim} CH^\bullet(X, m) \otimes \mathbb{Q}.$$

Both of the objects, higher K-theory and higher Chow groups are complicated to compute. Beilinson (via his $K_m(X)$) [2] and Bloch [3] (via his $CH^\bullet(X, m)$) constructed maps into computable cohomology theories. Such maps are called regulators. In this chapter, we will describe the real regulator from the higher Chow group $CH^k(X, m; \mathbb{Q})$ into real Deligne cohomology $H^{2k-m}_{\mathcal{D}}(X, \mathbb{R}(k))$ While presenting the subject, we will try to emphasize the similarities with the classical Chow groups and the cycle class maps.

3.1 Higher Chow Groups

Let X be a quasiprojective variety defined over a ground field k. For $n \in \mathbb{N}$ we define standard n-simplex by

$$\Delta^n = Spec(k[t_0, \cdots, t_n]/(\sum t_i - 1)).$$

Observe that Δ^n is a hyperplane in $\mathbb{A}_k^{n+1}$ defined by the equation $t_0 + \cdots + t_n = 1$, so is isomorphic to $\mathbb{A}_k^n$. We define the codimension one faces of the standard n-simplex Δ^n by setting the coordinates $t_i = 0$.

By intersecting the codimension one faces we get other faces with codimension greater than one. For example a codimension $n - k$ face is obtained by taking intersection of k codimension one faces (i.e: setting $t_i = 0$ for $i \in I \subset \{1, \cdots, n\}$ with $|I| = k$) and it is isomorphic to Δ^{n-k}.

The Abelian group generated by codimension k subvarieties of X is denoted by $z^k(X)$. A codimension k-cycle Z of $X \times \Delta^n$ meets $X \times \Delta^n$ properly if every component of Z meets all faces of $X \times \Delta^n$ in codimension greater or equal to k for all $m < n$. We set

$$z^k(X, n) = \{Z \in z^k(X \times \Delta^n) | \text{Z meets X} \times \Delta^n \text{properly}\}$$

Note that $z^k(X, 0) = z^k(X)$. One can define an "algebraic" version of singular homology. Let $\partial_i : z^k(X, m) \to z^k(X, m-1)$ be the restriction to the i-th face operator (Remember that i-th face is given by setting $t_i = 0$). Then the operator $\delta = \sum_{i=0}^{n} (-1)^n \partial_i : z^k(X, m) \to z^k(X, m-1)$ satisfies the boundary condition $\delta^2 = 0$.

The homology of the complex $z^k(X, \bullet), \delta$ yields the higher Chow groups of X:

Definition.[Bloch] The nth homology group of the complex

$$\cdots \longrightarrow z^k(X, n+1) \longrightarrow z^k(X, n) \longrightarrow z^k(X, n-1) \longrightarrow \cdots$$

is called the nth higher Chow group of X in codimension k and is denoted by $CH^k(X, n)$.

Alternatively one can define higher Chow groups using cubes instead of simplices which makes calculations easier in certain cases. Let $\square_k^n := (\mathbb{P}_k^1 \backslash \{1\})^n$ be the standard n-cube with coordinates z_i. The codimension one faces are obtained by setting the coordinates $z_i = 0, \infty$. Let ∂_i^0 and ∂_i^∞ denote the restriction maps to the faces $z_i = 0$ and $z_i = \infty$ respectively, then the boundary maps are given by;

$$\partial = \sum (-1)^{i-1} (\partial_i^0 - \partial_i^\infty).$$

Let $C^p(X, n)$ denote the free Abelian group generated by subvarieties of $X \times \square_k^n$ of codimension p meeting $X \times \square_k^n$ properly. Analogous to the simplicial case, we say a k-cycle of $X \times \square_k^n$ meets $X \times \square_k^n$ properly if every component of the cycle meets all faces $X \times \square_k^m$ of $X \times \square_k^n$ in codimension k for all $m < n$.

We have so called degenerate or decomposable cycles in the cubical version which we do not have in simplicial version. Notice that we have an isomorphism of varieties;

$$\square_k^{n-1} \times \square_k^1 \cong \square_k^n$$

.

Let $D^p(X, n)$ be the group (of degenerate cycles) generated by cycles which are pull backs of some cycles on $X \times \square_k^{n-1}$ coming from the standard projection of the n cube to the $n-1$ cube given by $(z_1, \cdots, z_n) \mapsto (z_1, \cdots, \hat{z}_i, \cdots, z_n)$ for some $i \in \{1, \cdots, n\}$.

Let $Z^p(X, \bullet)_{cub} := C^p(X, \bullet)/D^p(X, \bullet)$, then the higher Chow groups are defined to be the homology of the complex $(Z^p(X, \bullet)_{cub}, \partial)$.

It can be shown that the cubical and simplex versions of definitions of higher Chow groups coincide, because the complexes $Z^p(X, \bullet)$ and $Z^p(X, \bullet)_{cub}$ are known to be quasi-isomorphic.

There is also a characterization of elements in higher Chow groups, which is especially useful for writing the elements lying in $CH^n(X, 1)$ in explicit form, which comes from the Gerstein-Milnor resolution of a sheaf of Milnor K-groups.

For a field $\mathbb{F}$, the first two Milnor K-groups ([29]) are easy to characterize, namely $K_0(\mathbb{F}) = \mathbb{Z}$, $K_1(\mathbb{F}) = \mathbb{F}^\times$ and $K_2(\mathbb{F}) = \{(\mathbb{F}^\times \otimes_{\mathbb{Z}} \mathbb{F}^\times)/ \text{ Steinberg relations}\}$ where Steinberg relations are given as follows:

For $a, b \in \mathbb{F}^\times$,

$$\{a_1a_2, b\} = \{a_1, b\}\{a_2, b\}$$

$$\{a, b\} = \{b, a\}^{-1}$$

$$\{a, 1-a\} = \{a, -a\} = 1$$

We are interested in studying the higher Chow groups $CH^k(X, 1)$ so we are not going to define higher Chow groups $CH^k(X, n)$, $n \geq 2$, but similar argument we are going to provide works for them also.

One has a Gersten-Milnor resolution of a sheaf of Milnor K-groups on X ([16](p. 199)), whose last three terms are:

$$\cdots \longrightarrow \bigoplus_{cd_X Z = n-2} K_2^M(\mathbb{C}(Z)) \xrightarrow{T} \bigoplus_{cd_X Z = n-1} \mathbb{C}(Z)^\times \xrightarrow{div} \bigoplus_{cd_X Z = n} \mathbb{Z}$$

where div is the classical divisor map (zeros minus poles of a rational function) and T is the Tame symbol map.

The Tame symbol map is defined as follows:

Remember that

$$K_2^M(\mathbb{C}(Z)) = \{(\mathbb{C}(Z)^\times \otimes_{\mathbb{Z}} \mathbb{C}(Z)^\times)/\text{Steinberg relations and}$$

$$T : \bigoplus_{cd_X Z = n-2} K_2^M(\mathbb{C}(Z)) \longrightarrow \bigoplus_{cd_X Z = n-1} \mathbb{C}(Z)^\times$$

Let $\{f,g\} \in \mathbb{C}(Z)^{\times} \otimes_{\mathbb{Z}} \mathbb{C}(Z)^{\times}$

$$T(\{f,g\}) = \sum_{D} (-1)^{\nu_D(f)\nu_D(g)} (\frac{f^{\nu_D(g)}}{g^{\nu_D(f)}})_D$$

where $(\cdots)_D$ means restriction to the generic point of D and $\nu_D(f)$ is the order of vanishing of a rational function f along D.

The homology of the Gersten-Milnor resolution gives us the higher Chow groups $CH^r(X,m)$ in the case $m = 0,1,2$ From this identification, the higher Chow groups can be characterized as;

- $CH^n(X,0)$ is the free Abelian group generated by codimension n subvarieties in X modulo the divisors of rational functions on subvarieties of codimension $n-1$ in X. This is exactly the definition of classical Chow group $CH^n(X)$, so $CH^n(X) := CH^n(X,0)$.
- $CH^n(X,1)$ is represented by cycles of the form $\zeta = \sum_j (f_j, D_j)$ where $codim_X(D_j) = n-1$, $f_j \in \mathbb{C}(D_j)^{\times}$ and $\sum div(f_j)_{D_j} = 0$ modulo the image of the Tame symbol.
- $CH^n(X,2)$ is represented by classes in the kernel of the Tame map, modulo the image of a higher Tame symbol map.

3.1.1 Properties of Higher Chow Groups [31]

- The higher Chow groups $CH^{\bullet}(X,\bullet)$ are covariant for proper maps and contravariant for flat maps. ([3])
- For X smooth,we get a product structure using composition with pull back along the diagonal $X \to X \times X$([3]):

  $$CH^p(X,n) \otimes CH^q(X,m) \longrightarrow CH^{p+n}(X,n+m).$$

- Let X be a $k-$scheme, then

  $$CH^{\bullet}(X,n) \simeq CH(X \times \mathbb{A}^1_k, n).$$

  This property is called homotopy invariance property.

- As in the case of classical Chow groups, we have a localization sequence: Let $W \subset X$ be a closed subvariety of pure codimension r, then the localization sequence is ([3]):

$$\cdots \to CH^{\bullet - r}(W, n) \longrightarrow CH^{\bullet}(X, n) \longrightarrow CH^{\bullet}(X - W, n) \longrightarrow \cdots$$

- A very important property of higher Chow groups is the Riemann-Roch theorem for higher Chow groups proved by S. Bloch [3]. Let X be a smooth quasi-projective variety defined over k. Then there exists Chern maps $c_{n,p}^{Chow} : K_n(X) \to CH^p(X, n)$ and these maps induce an isomorphism called the Chern character map:

$$ch_n : K_n(X) \otimes \mathbb{Q} \simeq \bigoplus_{p \geq 0} CH^p(X, n) \otimes \mathbb{Q},$$

3.2 Deligne Cohomology

In the previous section, we have defined higher Chow groups as a natural generalization of classical Chow groups. The higher Chow ring $\bigoplus CH^p(X, n) \otimes \mathbb{Q}$ is isomorphic to the higher K-theory, $K_n(X) \otimes \mathbb{Q}$ by Bloch's version of the Grothendieck-Riemann-Roch theorem. Both the higher Chow ring, $\bigoplus CH^p(X, n) \otimes \mathbb{Q}$, and the higher K-theory of X,$K_n(X) \otimes \mathbb{Q}$, are complicated and it is difficult to compute these rings. We aim to construct a map to a "more computable" cohomology theory to study these complicated objects. For this purpose we will provide the definition of Deligne cohomology over rationals and real numbers. For a detailed discussion of Deligne cohomology see [14].

Let $\mathbb{A} \subset \mathbb{R}$ be a subring. For r a an integer, we put $\mathbb{A}(r) = (2\pi i)^r \mathbb{A}$. ($\mathbb{A}(r)$ is a pure Hodge structure of weight $-2r$ and type $(-r, -r)$). The Deligne complex is defined as:

$$\mathbb{A}_{\mathcal{D}}(r) := \mathbb{A}(r) \to \mathcal{O}_X \to \Omega_X \to \cdots \to \Omega_X^{r-1}.$$

For simplicity we will use the notation;

$$\Omega_X^{\bullet \leq r-1} := \mathcal{O}_X \to \Omega_X \to \cdots \to \Omega_X^{r-1}$$

which is De Rham complex cut at level $r-1$.

Definition 3.2.1. *Deligne cohomology is defined as the hypercohomology of the Deligne complex;*

$$H^i_{\mathcal{D}}(X, \mathbb{A}(r)) = \mathbb{H}^i(\mathbb{A}_{\mathcal{D}}(r))$$

Let us also recall what hypercohomology is. For a bounded sheaf complex $(\mathcal{F}^\bullet, d)$ on X and an open cover $\mathcal{U}$ of X, one has a Čech double complex;

$$(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}^\bullet), d, \delta).$$

One can construct an associated single complex;

$$\mathcal{S} := \bigoplus_{i+j=\bullet} \mathcal{C}^i(\mathcal{U}, \mathcal{F}^j) \qquad D = d \pm \delta.$$

The k-th hypercohomology is defined to be the k-th total cohomology of this associated single complex;

$$\mathbb{H}^k(\mathcal{F}^\bullet) := \varinjlim_{\mathcal{U}} H^k(\mathcal{S}^\bullet).$$

There are two filtered subcomplexes of the associated single complex $(\mathcal{S}, D)$; whose Grothendieck spectral sequences converges to $\mathbb{H}^k(\mathcal{F}^\bullet)$. For $p+q=k$;

$$'E_2^{p,q} := H^p_\delta(X, \mathcal{H}^q_d(\mathcal{F}^\bullet)$$

$$''E_2^{p,q} := H^p_d(\mathcal{H}^q_\delta(X, \mathcal{F}^\bullet)$$

The first spectral sequence reveals that complexes having same cohomology, i.e quasiisomorphic complexes, yield the same hypercohomology. This property allows us to give an alternative definition for Deligne cohomology.

Let $f : (A^\bullet, d) \longrightarrow (B^\bullet, d)$ be a morphism of complexes. The the cone complex

$$Cone(A^\bullet \xrightarrow{f} B^\bullet)$$

is defined as follows.

The q-th object is defined as

$$[Cone(A^\bullet \xrightarrow{f} B^\bullet)]^q = A^{q+1} \bigoplus B^q$$

and the differential is given by

$$\delta(a,b) := (-da, fa + db)$$

Consider the cone

$$Cone(\mathbb{A}(r) \bigoplus F^r\Omega_X^\bullet) \xrightarrow{\epsilon - l} \Omega^\bullet[-1]$$

where $[-1]$ means shifting all the terms in complex one position to the left. By definition, this cone complex is given by

$$\mathbb{A}(r) \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^{r-2} \xrightarrow{(0,d)} (\Omega_X^r \bigoplus \Omega_X^{r-1})$$
$$\xrightarrow{\delta} (\Omega_X^{r+1} \bigoplus \Omega_X^r) \xrightarrow{\delta} \cdots \xrightarrow{\delta} (\Omega_X^d \bigoplus \Omega_X^{d-1}) \longrightarrow \Omega_X^d$$

By holomorphic Poincare lemma, the natural map;

$$\mathbb{A}_{\mathcal{D}}(r) \to Cone(\mathbb{A}(r) \bigoplus F^r\Omega_X^\bullet) \xrightarrow{\epsilon - l} \Omega^\bullet[-1]$$

is a quasiisomorphism, hence both complexes yield the isomorphic Deligne cohomologies;

$$H^i_{\mathcal{D}}(X, \mathbb{A}(r)) \simeq \mathbb{H}^r(Cone(\mathbb{A}(r) \bigoplus F^r\Omega_X^\bullet) \xrightarrow{\epsilon - l} \Omega^\bullet[-1]).$$

We will work with real Deligne cohomology, so let us explore some frequently used sheaf complexes in Hodge theory, and their relations with Deligne cohomology.

Let $\mathbb{A}(r)$ be the constant sheaf, identified with the complex;

$$\mathbb{A}(r) \to 0 \to \cdots \to 0,$$

$\mathcal{D}_X^\bullet$ be the sheaf of currents on $C^\infty(2d-\bullet)$ forms and $\mathcal{C}_X^\bullet(\mathbb{A}(r))$ be the sheaf of Borel-Moore chains of real codimension $\bullet$. Among these sheaves some yield the same cohomology, we have the following quasiisomorphisms;

$$\mathbb{A}(r) \xrightarrow{\simeq} \mathcal{C}_X^\bullet(\mathbb{A}(r))$$
$$\Omega^\bullet \xrightarrow{\simeq} \mathcal{E}_X^\bullet$$
$$\mathcal{E}_X^\bullet \xrightarrow{\simeq} \mathcal{D}_X^\bullet$$

Note that the sheaves $\mathcal{C}_X^\bullet(\mathbb{A}(r)), \mathcal{E}_X^\bullet$ and $\mathcal{D}_X^\bullet$ are acyclic. Moreover the last two quasiisomorphisms are Hodge filtered. Using the quasiisomorphism above, we can rewrite the isomorphism

$$H^i_{\mathcal{D}}(X, \mathbb{A}(r)) \simeq \mathbb{H}^r(Cone(\mathbb{A}(r) \bigoplus F^r\Omega_X^\bullet) \xrightarrow{\epsilon - l} \Omega^\bullet[-1])$$

as

$$H^i_{\mathcal{D}}(X, \mathbb{A}(r)) \simeq \mathbb{H}^r(Cone(\mathcal{C}_X^\bullet(\mathbb{A}(r) \bigoplus F^r\ \mathcal{D}_X^\bullet) \xrightarrow{\epsilon - l} \mathcal{D}_X^\bullet[-1]).$$

The Hodge filter on these sheaf complexes allows us to express hypercohomology of these complexes in terms of filtered De Rham cohomology

$$\mathbb{H}^k(F^p\Omega_X^\bullet) \simeq \mathbb{H}^k(F^p\mathcal{E}_X^\bullet) \simeq F^pH^k_{dr}(X).$$

The hypercohomology of the De Rham complex cut out at level p can also be expressed in terms of the pieces of the De Rham complex;

$$\mathbb{H}(\Omega_X^{\bullet<p}) \simeq \frac{H^k_{dr}(X)}{F^pH^k_{dr}(X)}$$

The short exact sequence :

$$0 \to \Omega_X^{\bullet<k} \to \mathbb{A}_{\mathcal{D}}(k) \to \mathbb{A}(k) \to 0$$

induces the long exact sequence:

$$\cdots \to H^{2k-2}(X, \mathbb{Q}(k)) \to H^{2k-2}(X, \mathbb{C})/F^kH^{2k-2}(X, \mathbb{C}) \to H^{2k-1}_{\mathcal{D}}(X, \mathbb{Q}(k))$$
$$\xrightarrow{\alpha} H^{2k-1}(X, \mathbb{Q}(k)) \xrightarrow{\beta} H^{2k-1}(X, \mathbb{C})/F^kH^{2k-1}(X, \mathbb{C}) \to \cdots \quad (3.2.1)$$

There are no rational $2k-1$ classes in $F^kH^{2k-1}(X,\mathbb{Q}(k))$ so $ker(\beta)=0$ and in general we get a short exact sequence

$$0 \to \frac{H^{i-1}(X,\mathbb{C})}{H^{i-1}(X,\mathbb{A}(r))+F^rH^{i-1}(X,\mathbb{C})} \to H^i_{\mathcal{D}}(X,\mathbb{A}(r))$$
$$\to H^i(X,\mathbb{A}(r))\cap F^rH^i(X,\mathbb{C}) \to 0.$$

When $\mathbb{A}=\mathbb{Q}$ and $i=2k-1$, then we get the isomorphism

$$\begin{aligned} H^{2k-1}_{\mathcal{D}}(X,\mathbb{Q}(k)) &\simeq \frac{H^{2k-2}(X,\mathbb{C})}{F^kH^{2k-2}(X,\mathbb{C})+H^{2k-2}(X,\mathbb{Q})} \\ &\simeq \frac{(F^{d-k+1}H^{2d-2k+2}(X,\mathbb{C})^{\vee})}{H_{2d-2k+2}(X,\mathbb{Q}(d-k))} \end{aligned} \tag{3.2.2}$$

Next if we choose $\mathbb{A}=\mathbb{R}$, set $\mathbb{C}=\mathbb{R}(k)\oplus\mathbb{R}(k-1)$ and let π_{k-1} be the projection of $\mathbb{C}$ onto $\mathbb{R}(k-1)$. Then the isomorphism (3.2.2) decomposes through the map π_{k-1} and we get;

$$\begin{aligned} H^{2k-1}_{\mathcal{D}}(X,\mathbb{R}(k)) &\simeq \frac{H^{2k-2}(X,\mathbb{C})}{F^kH^{2k-2}(X,\mathbb{C})+H^{2k-2}(X,\mathbb{R})} \\ \xrightarrow[\simeq]{\pi_{k-1}} H^{k-1,k-1}&(X,\mathbb{R})\otimes\mathbb{R}(k-1) =: H^{k-1,k-1}(X,\mathbb{R}(k-1)) \\ &\simeq \{H^{n-k+1,n-k+1}(X,\mathbb{R}(n-k+1)\}^{\vee} \end{aligned} \tag{3.2.3}$$

3.3 The Real Regulator and Indecomposable Higher Chow Cycles.

We have defined Deligne cohomology and the higher Chow groups in the two previous sections. Higher Chow groups are a natural generalization of classical Chow groups, and working over reals or rationals Deligne cohomology can be expressed in terms of the de Rham cohomology up to a twist. We are ready to define the link between these objects. As in the classical case one can define a cycle class map. In the classical case the relation between algebraic cycles and De Rham cohomology was defined in terms of integration of differential forms which represents cohomology classes over the smooth parts of the algebraic varieties which constitutes the algebraic cycles. Although the cycle class and Abel Jacobi maps for higher Chow groups can be defined in general, we will discuss cycle class map only for the case of rational $K_1(X)$. For this special case the cycle class map has a simpler form which can be seen as a generalization of the classical case. We will deal with the real regulator for the rest of this study, so we will cook up the formulas and relations only in this case. For detailed presentation of cycle class and Abel-Jacobi maps for higher Chow groups, one can consult [23] and [24].

Let us define the cycle class map, or Chern class map, for the higher Chow groups

$$c_{k,1} := CH^k(X, 1 : \mathbb{Q}) \to H^{2k-1}_{\mathcal{D}}(X, \mathbb{Q}(k))$$

For a given higher Chow cycle $\zeta = \sum_i (Z_i, f_i) \in CH^k(X, 1 : \mathbb{Q})$, let $\gamma_i = f_i^{-1}[0, \infty]$ then $div(f_i) = \partial\gamma_i$ and let $\gamma = \sum_i \gamma_i$. Since $\zeta \in CH^k(X, 1 : \mathbb{Q})$, $\sum_i div(f_i) = 0$ which implies that $\partial\gamma = 0$. Hence γ defines a class in $H^{2k-1}(X, \mathbb{Q})$. Consider the long exact sequence (3.2.1). Up to a twist γ lies in the kernel of β, so γ bounds a $(2d - 2k + 2)$ chain ξ. Choosing the branch of logarithm along the $[0, \infty]$,one can define the current:

$$c_{k,1}(\xi) : \omega \to \frac{1}{(2\pi\sqrt{-1})^{d-k+1}} \left(\sum_i \int_{Z_i \setminus \gamma_i} \omega log(f_i) + 2\pi\sqrt{-1} \int_\xi \omega \right)$$

Considering the isomorphism (3.2.2), the current $c_{k,1}(\xi)$ defines the class of ξ in $H^{2k-1}_{\mathcal{D}}(X, \mathbb{Q}(k))$

If we consider the real coefficient Deligne cohomology, under the isomorphism (3.2.3) we get a current:

$$r_{k,1}(\xi) = \frac{1}{(2\pi\sqrt{-1})^{d-k+1}} \left(\sum_i \int_{Z_i \setminus Z_i^{sing}} \omega log|f_i| \right)$$

We will refer to $r_{k,1}(\xi)$ as the real regulator.

The image of the intersection product $CH^1(X, 1) \otimes CH^{k-1}(X)$ lies in $CH^k(X, 1)$. It is well known that for a field $\mathbb{F}$, $CH^1(X, 1) \simeq \mathbb{F}^\times$, so $CH^k(X, 1) \simeq \mathbb{C}^\times$. This image of the intersection product can be considered as the image of the classical Chow group $CH^{k-1}(X)$, up to a constant, in the higher Chow group $CH^k(X,1)$ and it is called group of decomposable cycles, denoted by $CH^k_{dec}(X,1)$. Considering the Gersten-Milnor resolution, the higher Chow cycle $\gamma \in CH^k(X,1)$ is represented as a formal sum $\gamma = \sum(g_j, Z_j)$ of nonzero rational functions g_j defined on irreducible subvarieties Z_j of codimension $k-1$ in X, such that $\sum div(g_j) = 0$. With this definition, decomposable cycles correspond to those with constant functions $g_j \in \mathbb{C}^\times$. The group of indecomposable cycles, denoted by $CH^k_{ind}(X,1)$, is defined to be the corresponding quotient

$$CH^k_{ind}(X,1) := \frac{CH^k(X,1)}{CH^k_{dec}(X,1)}$$

Can we use regulator maps to detect indecomposable higher Chow cycles? The answer to this question is positive and such a method formulated in terms of regulator indecomposable cycles is introduced in [17].

A higher Chow cycle $\zeta = \sum(g_j, Z_j)$ is called regulator indecomposable if the current defined by its real regulator

$$r(\zeta)(\omega) = \frac{1}{(2\pi\sqrt{-1})^{d-k+1}} \sum \left(\int_{Z_j - Z_j^{sing}} \omega \log|f| \right)$$

is nonzero for some test form $\omega \in (Hg^1(E_1 \times E_2 \otimes \mathbb{R}))^\perp$.

Let ξ be a decomposable higher Chow cycle, hence it is represented as $\sum_i (Y_i, f_i)$ with $Y_i \in CH^{k-1}(X)$ and $f_i \in \mathbb{C}^\times$. For any test form $\omega \in (Hg^1(E_1 \times E_2 \otimes \mathbb{R}))^\perp$, the regulator image;

$$r(\xi)(\omega) = \frac{1}{(2\pi\sqrt{-1})^{d-k+1}} \sum \left(\int_{Z_j - Z_j^{sing}} \omega \log|f| \right) \tag{3.3.1}$$

$$= \frac{1}{(2\pi\sqrt{-1})^{d-k+1}} \sum c_i \left(\int_X c_{k-1}(Z) \wedge \omega \right) = 0 \tag{3.3.2}$$

Hence a regulator indecomposable higher Chow cycle is indecomposable.

In next chapter, we are going to employ this method to prove that the group of indecomposable cycles is nontrivial for a sufficiently general product of two elliptic curves.

Chapter 4

An Indecomposable Higher Chow Cycle on a Product of Two Elliptic Curves

In the literature there are a number of results centered around proving that the group of indecomposable higher Chow cycles is nontrivial for certain algebraic varieties and constructing indecomposable cycles if possible. Some examples are [4, 8, 9, 12, 13, 27, 28, 30, 35]. Another subject of interest in this field is the structure of the group of indecomposable cycles; whether it is countably generated or not, whenever it is non-trivial. C. Voisin, [37] conjectured that the group of indecomposable cycles $CH^2_{ind}(X,1) \otimes \mathbb{Q}$ is countable for a smooth projective surface X. Actually there are no Hodge theoretic obstructions to countability of $CH^2_{ind}(X,1)$ for such varieties. An example of a countably infinitely generated group of indecomposable cycles is given by A.Collino. In [9], he proves that the group of indecomposable cycles $CH^3_{ind}(X,1) \otimes \mathbb{Q}$ is countably infinitely generated for a general cubic fourfold X.

Geometrically rich and well understood varieties are natural candidates in which one can construct indecomposable higher Chow cycles. Families of products of curves, K3 and Kummer surfaces and their deformations have widely been studied. One such result in this direction is the following theorem presented in [17];

Theorem 4.0.1 (Theorem 1)**.** *When $X = E_1 \times E_2$ is a sufficiently general product of two elliptic curves, then $CH^2_{ind}(X, 1) \otimes \mathbb{Q} \neq 0$,i.e there exists a nontrivial indecomposable higher Chow cycle ξ on X.*

The prove this statement, a regulator indecomposable higher Chow cycle is constructed using the geometry of the elliptic curves and considering the deformations of families of such varieties X. Together with the results in [27], this theorem provides stronger results on the nature of indecomposables [17];

Theorem 4.0.2 (Theorem 2)**.** *When $X = E_1 \times E_2 \times E_3$ is a sufficiently general product of three elliptic curves, then the level of $CH^3_{ind}(X, 1) \otimes \mathbb{Q}$, is at least 1.*

As a corollary of this theorem it follows that

Corollary 4.0.3. *[17][Corollary 1] When $X = E_1 \times E_2 \times E_3$ is a sufficiently general product of three elliptic curves, then $CH^3_{ind}(X, 1) \otimes \mathbb{Q}$, is uncountable.*

It is shown by M. Saito that the cycle constructed in [17] is in fact decomposable contrary to the claim. However the results presented in [17] are valid and were proved by totally different techniques later. In [6] the Hodge-$\mathcal{D}$ conjecture for surfaces of the form $E_1 \times E_2$, where E_1 and E_2 are general elliptic curves and for general Abelian varieties is proved. Theorem 1 of [17] follows from that result.

The motivation and starting point of this thesis was to recover the results presented in [17] following its spirit. We have been able to prove Theorem 1, constructing a regulator indecomposable higher Chow cycle [36].

In the first section we will discuss the error in [17]. In the second section, we will explain the construction of the regulator indecomposable higher Chow cycle given in [36]. The consequences of this result and further possible research is discussed in the last section of this chapter.

4.1 The Setting and the Error

Before we start discussing the ideas presented in [17] we want to fix some notation and conventions. For a subring $\mathbb{A} \subset \mathbb{R}$, put $\mathbb{A}(k) = \mathbb{A}(2\pi\sqrt{-1})^k$. For the higher Chow groups $CH^k X, m$ and for $\mathbb{A}$ as above, we denote $CH^k(X, m) \otimes \mathbb{A}$ by $CH^k(X, m; \mathbb{A})$. Methods presented here factor through rational and real Deligne cohomology which is blind to torsion, it is convenient to work with $CH^k(X, m; \mathbb{Q})$. Finally, "sufficiently general X" means, $X = X_t$ with t outside a suitable countable union of Zariski closed subsets. Our notation will be compatible with [17].

We will start with describing the setting of [17]. We will be working on a product of two elliptic curves $E_1 \times E_2$, so we begin with defining the coordinates on the ambient space $\mathbb{P}^2 \times \mathbb{P}^2$. Let $[s_0, s_1, s_2]$ be the homogeneous coordinates on the first copy and $[t_0, t_1, t_2]$ on the second copy of $\mathbb{P}^2$ and correspondingly $(x_1, y_1) = (s_1/s_0, s_2/s_0)$ and $(x_2, y_2) = (t_1/t_0, t_2/t_0)$ be the affine coordinates.

We can define elliptic curves in terms of non-singular cubic polynomials. Let $E_j = V(\overline{F_j}) \subset \mathbb{P}^2$ where $\overline{F_j}$ is the homogenization of the Weierstrass equation $F_j = y_j^2 - x_j^3 + b_j x_j + c_j$ with nonzero discriminant $\Delta_j = 4b_j^3 + 27c_j^2 \neq 0$ for $j = 1, 2$. In terms of the corresponding homogeneous coordinates, E_j's are given by the equations;

$$F_1 := s_0 s_2^2 - s_1^3 - b_1 s_0^2 s_1 - c_1 s_0^3$$
$$F_2 := t_0 t_2^2 - t_1^3 - b_2 t_0^2 t_1 - c_2 t_0^3$$

We define X to be product of two elliptic curves $X := V(\overline{F_1}, \overline{F_2}) \simeq E_1 \times E_2$ Now, let $\overline{F_0} = s_1 t_1 + s_2 t_2$ and let $D := V(\overline{F_0}, \overline{F_1}, \overline{F_2})$; be the intersection of $E_1 \times E_2$ with the hyperplane defined by $V(\overline{F_0})$, hence D can be thought of as a curve lying in $E_1 \times E_2$.

Observe that given $t = (b_1, c_1, b_2, c_2) \in \mathbb{C}^4$ determines X. Hence we can consider the family given by

$$\mathcal{X} := V(\overline{F_1}, \overline{F_2}) \subset \mathbb{C}^4 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{C}^4$$

A "Sufficiently general product of two elliptic curves X" means, $X = X_t$ with

t outside a suitable countable union of Zariski closed subset of the base space $\mathbb{C}^4$.

For sufficiently general X, D is smooth and irreducible (see [17],[Lemma 2.2]).

The first step is to construct an indecomposable higher Chow cycle is to start with a tuple (f, D), where D is defined above and $f = x_1 + \sqrt{-1}$. Since f is a linear function defined only on E_1 ,graph of f intersects E_1 at three points, (one of them is the point at the infinity). Using the additive structure of elliptic curves and Abel's theorem, the tuple (f, D) is completed to a higher Chow cycle $\xi = (f, D) + \sum(C_i, g_i)$ where C_i's are curves supported in either E_1 or E_2 and $g_i \in \mathbb{C}^\times(C_i)$ are functions such that $div_{E_1}(f_i) + \sum div_{C_i}(g_i) = 0$.

To prove that this higher Chow cycle ξ is indecomposable, the authors claim that the regulator image of ξ; $r(\xi)(\omega)$ is nonzero for a test form ω. The form $\omega := -2\pi\sqrt{-1}(\frac{dx_1}{y_1} \wedge \frac{d\overline{x}_2}{\overline{y}_2} + \frac{d\overline{x}_1}{\overline{y}_1} \wedge \frac{dx_2}{y_2})$ in affine coordinates is considered. Then for general X, $\omega \in (Hg^1(X) wehave \oplus \mathbb{R})^\perp$ (see Lemma 2.5 [17]).

Note that

$$r_{2,1}(\xi)(\omega) = \frac{1}{(2\pi\sqrt{-1})}\left(\int_D \omega log|f| + \sum_i \int_{C_i \setminus C_i^{sing}} \omega log|g_i|\right).$$

Since the curves C_i are supported in either in E_1 or E_2, they can not support the real two form ω given above. Hence

$$\int_{C_i \setminus C_i^{sing}} \omega log|g_i| = 0 \qquad \forall i$$

So these terms which are introduced to complete the tuple (f, D) to a higher Chow cycle, and do not contribute to the real regulator are called “degenerate terms”.

The only contribution to the real regulator comes from the tuple (f, D);

$$r_{2,1}(\xi)(\omega) = \frac{1}{(2\pi\sqrt{-1})}\int_D \omega log|f|.$$

It is claimed that this integral is non zero for sufficiently general X. This claim is proved by means of two deformation arguments. First, deforming D_t from the generic point $t = (b_1, c_1, b_2, c_2)$ to $t = (b_1, 0, b_2, 0)$ and then considering the limit case as $(b_1, b_2) \mapsto (0, 0)$. However there is an error in the second deformation argument of ,[17]. We would like to discuss this error briefly before we alter this problem in next section.

The following proposition describes how the curve D_t changes under the deformations, we will consider.

Proposition 4.1.1. *[17][Proposition 2.7]*
(1) If $t = (b_1, 0, b_2, 0)$, i.e; $h_j(x_j) = x_j^3 + b_j x_j$ for $j = 1, 2$, then

$$D = (E_1 \times [1, 0, 0]) + ([1, 0, 0] \times E_2) + \grave{D}$$

and x_1 is a local coordinate on nonempty Zariski-open subset of each irreducible component of $\grave{D}$
(ii) If $t = (0, 0, 0, 0)$, i.e; $h_j(x_j) = x_j^3$ for $j = 1, 2$, then

$$D = (E_1 \times [1, 0, 0]) + ([1, 0, 0] \times E_2) + \H{D}$$

where locally $\H{D}$ is described by

$$\H{D} = V(y_1^2 - x_1^3, y_2^2 - x_2^3, x_1x_2 + y_1y_2, x_1x_2 - 1)$$

In particular, $\H{D}$ is irreducible and x_1 is a local coordinate on a nonempty Zariski-open subset of $\H{D}$.

When $t = (b_1, 0, b_2, 0)$, we have $X = E_1 \times E_2$ where E_j is given by the equation $y_j^2 = x_j^3 + b_j x_j$ and $D_t = X \cap V(x_1x_2 + y_1y_2 = 0)$. Notice that on D_t we have

$$x_1^2 x_2^2 = y_1^2 y_2^2 = x_1 x_2 (x_1^2 + b_1)(x_2^2 + b_2)$$

and we can decompose

$$D_t = (E_1 \times [1, 0, 0]) + ([1, 0, 0] \times E_2) + \grave{D}_t$$

where $x_1 x_2 = (x_1^2 + b_1)(x_2^2 + b_2)$ on $\grave{D}_t$. We can cancel a factor of $x_1 x_2$ which corresponds to the curve $(E_1 \times [1, 0, 0]) + ([1, 0, 0] \times E_2)$ since the pull back of the real 2-form ω to this component is zero. Hence we have

$$\int_{D_t} \omega \log |f| = \int_{\grave{D}_t} \omega \log |f|$$

and we are left with the family $\sum := \bigcup_{t\in U} \grave{D}_t$ for some neighbourhood U of t.

In the second degeneration argument; $(b_1, b_2) \mapsto (0,0)$, we have $X = E_1 \times E_2$, where the elliptic curves E_j themselves degenerate to $y_j^2 = x_j^3$ and we can decompose $\grave{D}_t$ into three pieces $\H{D}$, $(E_1 \times [1,0,0])$ and $([1,0,0] \times E_2)$ where $\H{D} = D \cap V(x_1x_2 - 1)$. Moreover, we have $x_1x_2 = x_1^2x_2^2$ on $\H{D}$, but this time we can not cancel the factor x_1x_2, since the real 2-form ω acquires singularities and contributions to the real regulator from different parts cancel each other.

We will keep track of this deformation and show that the contributions to real regulator from the parts $\H{D}$ and $(E_1 \times [1,0,0])$ cancel each other by direct calculation of integrands in the limit case. To see this, and for notational simplicity, let us take $b_1 = b_2 = \epsilon$. On $\grave{D}$, we have $x_1x_2 = (x_1^2 + \epsilon)(x_2^2 + \epsilon)$ and x_1 is a local coordinate on a Zariski open subset of each irreducible component of $\grave{D}$, (provided we discard the component $[1,0,0] \times E_2$ when $b_1 = b_2 = 0$, which we can do, as this amounts to the observation that $\log|f| = \log|x_1 - \sqrt{-1}| = 0$ there). We now apply some first order approximations. For small values of $|\epsilon|$, we have $x_1x_2 \approx x_1^2x_2^2$ and if $x_1x_2 \neq 0$, then $x_1x_2 = 1$, and $x_2 \approx x_1^{-1}$ is a solution. On the other hand regarding $E_1 \times [1,0,0]$, we look at small values of $|x_2|$, and we get $x_1x_2 \approx \epsilon(x_1^2 + \epsilon) \approx \epsilon x_1^2$, and $x_2 \approx \epsilon x_1$ is a solution. Clearly, the former one limits to $\H{D}$ and the latter to $E_1 \times [1,0,0]$. Reiterating, we can discard the other component $[1,0,0] \times E_2$. So we will compute the limiting integral of $\log|x_1 - \sqrt{-1}|\omega$ for these two approximate solutions.

Consider

$$\omega = \left(\frac{dx_1}{\sqrt{x_1^3 + \epsilon x_1}}\right) \wedge \overline{\left(\frac{dx_2}{\sqrt{x_2^3 + \epsilon x_2}}\right)} + \overline{\left(\frac{dx_1}{\sqrt{x_1^3 + \epsilon x_1}}\right)} \wedge \left(\frac{dx_2}{\sqrt{x_2^3 + \epsilon x_2}}\right). \quad (4.1.1)$$

For $x_2 = x_1^{-1}$, $dx_2 = -x_1^{-2}dx_1$. Plugging this into the equation above,

$$\omega = \left(\frac{dx_1}{(x_1^3 + \epsilon x_1)^{\frac{1}{2}}}\right) \wedge \overline{\left(\frac{-x_1^{-2}dx_1}{(x_1^{-3} + \epsilon x_1^{-1})^{\frac{1}{2}}}\right)} + \overline{\left(\frac{dx_1}{(x_1^3 + \epsilon x_1)^{\frac{1}{2}}}\right)} \wedge \left(\frac{-x_1^{-2}dx_1}{(x_1^{-3} + \epsilon x_1^{-1})^{\frac{1}{2}}}\right).$$

Arranging the terms, we get;

$$\begin{aligned}\omega &= -\frac{dx_1}{x_1^{\frac{1}{2}}(x_1^2+\epsilon)^{\frac{1}{2}}} \wedge \frac{d\overline{x_1}}{\overline{x_1}^{\frac{1}{2}}\overline{(1+\epsilon x_1^2)}^{\frac{1}{2}}} - \frac{d\overline{x_1}}{\overline{x_1}^{\frac{1}{2}}\overline{(x_1^2+\epsilon)}^{\frac{1}{2}}} \wedge \frac{dx_1}{x_1^{\frac{1}{2}}(1+\epsilon x_1^2)^{\frac{1}{2}}} \\ &= \left(\frac{-1}{x_1^{\frac{1}{2}}(x_1^2+\epsilon)^{\frac{1}{2}}\overline{x_1}^{\frac{1}{2}}\overline{(1+\epsilon x_1^2)}^{\frac{1}{2}}} + \frac{1}{\overline{x_1}^{\frac{1}{2}}\overline{(x_1^2+\epsilon)}^{\frac{1}{2}}x_1^{\frac{1}{2}}(1+\epsilon x_1^2)^{\frac{1}{2}}}\right) dx_1 \wedge d\overline{x_1} \\ &= \left(\frac{x_1^{\frac{1}{2}}(x_1^2+\epsilon)^{\frac{1}{2}}\overline{x_1}^{\frac{1}{2}}\overline{(1+\epsilon x_1^2)}^{\frac{1}{2}} - \overline{x_1}^{\frac{1}{2}}\overline{(1+\epsilon x_1^2)}^{\frac{1}{2}}x_1^{\frac{1}{2}}(1+\epsilon x_1^2)^{\frac{1}{2}}}{|x_1||1+\epsilon x_1^2||x_1^2+\epsilon||x_1|}\right) dx_1 \wedge d\overline{x_1}.\end{aligned}$$

Taking the limit as $\epsilon \to 0$,

$$\omega = \left(\frac{x_1^{\frac{3}{2}}\overline{x_1}^{\frac{1}{2}} - x_1^{\frac{1}{2}}\overline{x_1}^{\frac{3}{2}}}{|x_1|^4}\right) dx_1 \wedge d\overline{x_1} = \left(\frac{x_1 - \overline{x_1}}{|x_1|^3}\right) dx_1 \wedge d\overline{x_1} \text{ on } \H{D}.$$

In the limit as $\epsilon \to 0$, $x_2 = x_1^{-1}$ has limit $\H{D}$ and

$$\log|f|\omega \to \log|x_1 - \sqrt{-1}|\left(\frac{x_1 - \overline{x_1}}{|x_1|^3}\right) dx_1 \wedge d\overline{x_1}.$$

Let us consider the latter approximation $x_2 = \epsilon x_1$. When $x_2 = \epsilon x_1$; $dx_2 = \epsilon dx_1$, plugging these relations in Equation (4.1.1), we get;

$$\begin{aligned}\omega &= \left(\frac{dx_1}{(x_1^3+\epsilon x_1)^{\frac{1}{2}}}\right) \wedge \overline{\left(\frac{\epsilon dx_1}{(\epsilon^3 x_1^3+\epsilon^2 x_1)^{\frac{1}{2}}}\right)} + \overline{\left(\frac{dx_1}{(x_1^3+\epsilon x_1)^{\frac{1}{2}}}\right)} \wedge \left(\frac{\epsilon dx_1}{(\epsilon^3 x_1^3+\epsilon^2 x_1)^{\frac{1}{2}}}\right) \\ &= \left(\frac{dx_1}{(x_1^3+\epsilon x_1)^{\frac{1}{2}}}\right) \wedge \frac{\overline{\epsilon} d\overline{x_1}}{\overline{(\epsilon^3 x_1^3+\epsilon^2 x_1)^{\frac{1}{2}}}} + \frac{d\overline{x_1}}{\overline{(x_1^3+\epsilon x_1)^{\frac{1}{2}}}} \wedge \left(\frac{\epsilon dx_1}{(\epsilon^3 x_1^3+\epsilon^2 x_1)^{\frac{1}{2}}}\right) \\ &= \left(\frac{\overline{\epsilon}}{(x_1^3+\epsilon x_1)^{\frac{1}{2}}\overline{(\epsilon^3 x_1^3+\epsilon^2 x_1)^{\frac{1}{2}}}} - \frac{\epsilon}{\overline{(x_1^3+\epsilon x_1)^{\frac{1}{2}}}(\epsilon^3 x_1^3+\epsilon^2 x_1)^{\frac{1}{2}}}\right) dx_1 \wedge d\overline{x_1}.\end{aligned}$$

Taking the limit as $\epsilon \to 0$, we get,

$$\omega = \left(\frac{1}{x_1^{\frac{3}{2}}\overline{x_1}^{\frac{1}{2}}} - \frac{1}{\overline{x_1}^{\frac{3}{2}}x_1^{\frac{1}{2}}}\right) dx_1 \wedge d\overline{x_1} = \left(\frac{\overline{x_1} - x_1}{|x_1|^3}\right) dx_1 \wedge d\overline{x_1} \text{ on } E_1 \times [1,0,0].$$

In the limit as $\epsilon \to 0$, $x_2 = \epsilon x_1$ has limit $E_1 \times [1,0,0]$ and

$$\log|f|\omega \to \log|x_1 - \sqrt{-1}|\left(\frac{\overline{x_1} - x_1}{|x_1|^3}\right)dx_1 \wedge d\overline{x_1}.$$

(As a reminder, when $b_1 = b_2 = 0$, $E_1 = E_2$ are (singular) rational curves.) In the limit, the contributions of these parts to the real regulator cancel.

4.2 Constructing a Higher Chow Cycle

Remember that the error in [17] was due to the fact that the real 2-form acquired singularities in the degenerate case and the contributions to the real regulator from different parts canceled each other, resulting in a regulator decomposable cycle on the contrary to the claim. Even though the resulting cycle turns out to be decomposable, the method used is very natural and can be restored. The problem of acquiring singularities in the degenerate case can be altered by considering a slightly different function instead of the original one.

In order to solve this problem, we consider the function $f = x_1^2 x_2 - \sqrt{-1}$ and the same form ω. Note that for the solution $x_2 = \epsilon x_1$, which limits to the component $E_1 \times [1,0,0]$, $\log|x_1^2 x_2 - \sqrt{-1}| = \log|\epsilon x_1^3 - \sqrt{-1}|$ goes to zero as $\epsilon \to 0$, so in the limit $\log|f|\omega$ vanishes. However for the second solution $x_2 = x_1^{-1}$, we have $\log|x_1^2 x_2 - \sqrt{-1}| = \log|x_1 - \sqrt{-1}|$. In the limit we get the component $\tilde{\tilde{D}}$ and recover the function $\log|x_1 - \sqrt{-1}|$ introduced in [17], which contributes to the real regulator nontrivially.

Even though the solution to the error in [17] seems like a simple alteration, the real price of alteration is paid when the tuple (f, D) is completed to a higher Chow cycle. In [17], the function $f = x_1 + \sqrt{-1}$ is linear hence, the group law on elliptic curves and Abel's theorem enables one to easily find curves C_i and functions g_i on those curves such that $\xi = (f, D) + \sum(C_i, g_i)$ is a higher Chow cycle, i.e; $div_{E_1}(f) + \sum div_{C_i} g_i = 0$. The function we consider $f = x_1^2 x_2 - \sqrt{-1}$ is not linear and it requires more complicated considerations to complete the tuple (f, D) to a higher Chow cycle.

Consider the Segre embedding $s : \mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$, given by

$$s : [s_0, s_1, s_2; t_0, t_1, t_2] \mapsto [s_0t_0, s_1t_0, s_2t_0, s_0t_1, s_1t_1, s_2t_1, s_0t_2, s_1t_2, s_2t_2]$$

Under the Segre embedding, the curve D corresponds to a $\mathbb{P}^7 \subset \mathbb{P}^8$ intersecting with X. In projective coordinates the function f is given by

$$f = x_1^2x_2 + \sqrt{-1} = \frac{s_1^2t_1t_0 + s_0^2t_0^2\sqrt{-1}}{s_0^2t_0^2}.$$

Under the Segre embedding f is a quotient of two quadrics

$$Q_{1,0} = s_1^2t_1t_0 + s_0^2t_0^2\sqrt{-1} = (s_1t_1)(s_1t_0) + (s_0t_0)^2\sqrt{-1}$$

and

$$Q_{2,0} = s_0^2t_0^2 = (s_0t_0)^2.$$

Counted with multiplicities the divisor of f along D in $\mathbb{P}^8$ is given by

$$div(f)_D = V(Q_{1,0}) \cap D - V(Q_{2,0}) \cap D.$$

Now let $E_{j,tor}$ denote the set of torsion points on E_j. We define $D_{tor} := \{E_{1,tor} \times E_2\} \cap D$. For sufficiently general X, D is a smooth, irreducible curve. Moreover $E_{1,tor}$ is dense in E_1 and $D \subset X = E_1 \times E_2$ projects onto the first factor, so D_{tor} is dense in D.

A quadric $Q \in \mathbb{P}^8$ intersects D in 36 points up to multiplicity; deg $(Q \cap D) = 36$. Consider the family of quadrics lying in a $\mathbb{P}^7 \subset \mathbb{P}^8$ cutting out $D \in E_1 \times E_2$ under the Segre embedding. This family is a projective space of dimension 35, so the family of quadrics passing through 35 general points of D is zero dimensional. If we set $Q \cap D = \{p_1 + \cdots + p_{36}\}$, and suppose $\{p_1 \cdots p_{35}\} \in D_{tor}$, then $p_{36} \in D_{tor}$.

Let $q_1^i \cdots q_{36}^i \in div_D(Q_{i,0})$. Since D_{tor} is dense in D, for any given collection of neighborhoods $\{U_i\}$ around q_i for $i = 1 \cdots 36$, we can find 36 points $p_1^i, \cdots, p_{36}^i \in D_{tor}$ in general position such that $p_j^i \in U_i$. By the argument above these points define quadratic functions $Q_{i,n}$ for $i = 1, 2$ and $\tilde{f}_n = Q_{1,n}/Q_{2,n}$ such that $p_1^i, \cdots, p_{36}^i \in div_D(\tilde{f}_i) \subset D_{tor}$, moreover $\lim_{n\to\infty} \tilde{f}_n = f$.

We have started with a function f defined on D and shown that this function f can be continuously deformed to another function $\tilde{f}$ such that $div_D(\tilde{f}) \in D_{tor}$. We need such a deformation to complete the tuple $(\tilde{f}, D)$ to a higher Chow cycle, using the geometry of torsion points. Before we describe how this tuple can be completed to o higher Chow cycle, we want to note a property of the deformation we have considered which is very crucial for our construction. We claim that the integral

$$\int_D \log \tilde{f} \omega \neq 0.$$

In other words we can deform f to $\tilde{f}$ preserving the non triviality of the contribution to the real regulator.

Let Δ_j be an open polydisk in the space of quadratic polynomials in $\mathbb{C}[z_0, \cdots z_7]$ centered at 0 for $j = 1, 2$. Then for $t \in \Delta := \Delta_1 \times \Delta_2$, one has a corresponding function $f_t = Q_{1,t}/Q_{2,t}$ with $f_0 = f$.

Note that the set

$$\bigcup_{t \in \Delta} |div f_t|$$

has real codimension ≥ 2 in $\Delta \times D$. Considering tubular neighborhoods in $\Delta \times D$ and shrinking them if necessary we conclude that the integral

$$\int_D \log |f_t| \omega$$

varies continuously with $t \in \Delta$.

We may assume that

$$\int_D \log |f_t| \omega \neq 0, \qquad \forall t \in \Delta$$

for some polydisk Δ. Note that if $h_1, h_2 \in \mathbb{C}^\times$ with $div(h_1) = div(h_2)$ then $h_1 = c.h_2$ for some $c \in \mathbb{C}^\times$. By perturbing $t \in \Delta$, we can assume that up to a constant $Q_{1,0}$ and $Q_{2,0}$ are unique quadratics defining f. Since Δ parametrizes all quadratic quotients in a neighborhood of $(0,0) \in \Delta$, for large enough n we will have $\tilde{f}_n = f_t$ for some $t \in \Delta$. The integral varies continuously in the disk Δ, therefore we have

$$\left| \int_D \log |f| \omega - \int_D \log |\tilde{f}_n| \omega \right| < \epsilon$$

for some large enough n.

Now we are ready to complete the tuple $(\tilde{f}, D)$ to a higher Chow cycle. Note that the integral

$$\int_D \log|\tilde{f}_n|\omega \neq 0.$$

First observe that the divisor of $\tilde{f}$ along D can be written as;

$$div(\tilde{f})_D = \sum_j n_j(p_j \times q_j) \in D_{tor} \quad \text{where } \sum_j n_j = 0.$$

Let e_j denote the identity element on E_j. By our construction, the p_j's are torsion points so $m_j p_j \sim_{rat} m_j e_1$ for some m_j (there exist rational functions $h_j's \in \mathbb{C}(E_1)^\times$ such that $div_{E_1}(h_j) = m_j e_1 - m_j p_j$). Then for $m = gcd(\{m_j\}$, we have $mp_j \sim_{rat} me_1$ for all j. So we can find rational functions $h_j's \in \mathbb{C}(E_1 \times q_j)^\times$ such that $div_{E_1\times q_j}(h_j) = m(e_1 \times q_j) - m(p_j \times q_j)$. Consider the precycle $(\tilde{f}^m, D) + \{h_j^{n_j}, E_1 \times q_j\}_j$;

$$\begin{aligned} div_D(\tilde{f}^m) + \sum_j div_{E_1\times q_j}(h_j^{n_j}) &= \sum_j mn_j(p_j \times q_j) + \sum_j (mn_j(e_1 \times q_j) - mn_j(p_j \times q_j)) \\ &= \sum_j mn_j(e_1 \times q_j) := \xi \end{aligned}$$

The remaining term ξ is the divisors of the functions $\tilde{f}$ and $\{(h_j)\}_j$, hence it is rationally equivalent to zero on $E_1 \times E_2$. The projection of ξ to second factor, $\mathrm{Pr}_{2,*}(\xi)$, is rationally equivalent to zero on E_2. So there exists a rational function g defined on $e \times E_2$ such that $div_{e\times E_2}(g) = -\sum_j mn_j(e \times q_j)$. Let $\gamma = (\tilde{f}^m, D) + \{(h_j^{n_j}, E_1 \times q_j)\}_j + (g, e_1 \times E_2)$. Then

$$div_D(\tilde{f}^m) + \sum_j div_{E_1\times q_j}(h_j^{n_j}) + div_{e_1\times E_2}(g) = 0.$$

Hence $\gamma \in CH^2(X, 1; Q)$ is a higher Chow cycle. Note that the curves $E_1 \times q_j$ and $p_j \times E_2$ can not support the real 2-form ω. Therefore the contributions of the terms $\{(h_j^{n_j}, E_1 \times q_j)\}_j + (g, e \times E_2)$ to the real regulator are zero

$$\int_{E_1\times q_j} \log|h_j|\omega = 0 = \int_{e\times E_2} \log|g|\omega.$$

The only contribution to the real regulator comes from the tuple $(\tilde{f}^m), D$;

$$r(\gamma)(\omega) = \int_D \omega \log |\tilde{f}^m| \neq 0$$

i.e; $\gamma \in CH^2(X, 1; \mathbb{Q})$ is regulator indecomposable so it is indecomposable.

Theorem 4.2.1. $CH^2_{ind}(E_1 \times E_2, 1)$ *is nontrivial for a sufficiently general product* $E_1 \times E_2$ *of elliptic curves* E_1 *and* E_2.

4.3 Consequences, Implications and Possible Further Research

The proof of 4.2.1 we have provided in this thesis is in the same spirit as the paper it is first purposed [17]. This result follows from the main result of [6], which is proved by means of a totally different set of techniques.

In [6], Hodge-$\mathcal{D}$ conjecture is proved for general K3 surfaces and Abelian surfaces. Beilinson's Hodge-$\mathcal{D}$ conjecture for real varieties implies that the real regulator;

$$r_{k,1} \otimes \mathbb{R} : CH^k(X, 1) \otimes \mathbb{R} \to H^{2k-1}_{\mathcal{D}}(X, \mathbb{R}(k)) \text{ is surjective ([22]).}$$

For the case of sufficiently general product of two elliptic curves $X = E_1 \times E_2$ this implies that the reduced regulator;

$$CH^2_{ind}(X, 1) \otimes \mathbb{R} \longrightarrow H^3_{\mathcal{D}}(X, \mathbb{R}(2)) \longrightarrow H^{1,1}(X, \mathbb{R}(1))$$

is surjective.

The proof provided in [6] considers the degeneration of a general $K3$ surface to a $K3$ surface with maximum Picard number and uses rational curves to show the existence of indecomposable cycles.

The proof of 4.2.1 we give in this thesis, can also be used to provide a alternative proof for the Hodge-$\mathcal{D}$ conjecture for a sufficiently general product of two elliptic curves.

For a sufficiently general product of two elliptic curves, Beilinson's Hodge-$\mathcal{D}$ conjecture can be stated as

$$r_{2,1} \otimes \mathbb{R} : CH^2(E_1 \times E_2, 1) \otimes \mathbb{R} \to H^3_{\mathcal{D}}(X, \mathbb{R}(2)) \text{ is surjective.}$$

The Classical Hodge Conjecture holds for sufficiently general product of two elliptic curves, so real regulator is surjective on this part. It is enough to prove that the reduced regulator

$$CH^2_{ind}(X, 1) \otimes \mathbb{R} \longrightarrow H^{1,1}(X, \mathbb{R}(1))$$

is surjective.

The image has rank two and is generated by the forms

$$\omega_1 := 2\pi\sqrt{-1}\left(\frac{dx_1}{y_1} \wedge \frac{d\overline{x}_2}{\overline{y}_2} + \frac{d\overline{x}_1}{\overline{y}_1} \wedge \frac{dx_2}{y_2}\right)$$

$$\omega_2 := 2\pi\sqrt{-1}\left(i\frac{dx_1}{y_1} \wedge \frac{d\overline{x}_2}{\overline{y}_2} - i\frac{d\overline{x}_1}{\overline{y}_1} \wedge \frac{dx_2}{y_2}\right)$$

Notice that ω_1 is the test form we have used to prove that the higher Chow cycle γ is regulator indecomposable and $r(\gamma)(\omega_1) \neq 0$. If we can find a function g such that $r(\gamma)(\omega_2) \neq 0$ we will prove that the reduced real regulator is surjective. The function $g = x_1x_2^2 + 1$ satisfies this criterion. Currently I am working on the details of proof of this statement.

Once we construct a higher Chow cycle on a sufficiently general product two elliptic curves, the natural question is whether we can generalize this construction for product of more than two curves. Even though the idea of using geometry of elliptic curves and torsion points does not work in the case of three elliptic curves, our result 4.2.1 together with theorem of Rosenschon and Saito given below, provides a stronger statement about the group structure of indecomposable cycles for this case.

Theorem 4.3.1. *[33] Let X_1 and X_2 be smooth projective varieties and $X = X_1 \times X_2$. Assume that $H^{1,0}(X_1) \neq 0$ and the reduced regulator*

$$CH^{r-1}(X_2, 1) \to H^{2r-3}_{\mathcal{D}}(X_2, \mathbb{Q}(r-1)) \to$$

$$\frac{H^{2r-4}(X_2, \mathbb{C})}{F^{r-1}H^{2r-4}(X_2, \mathbb{C}) + H^{2r-4}(X_2, \mathbb{Q}(r-1)) + H^{r-2,r-2}(X_2, \mathbb{Q}(r-1)) \otimes \mathbb{C}}$$

is non-trivial. Then the image of the composite

$$Pic^0(X_1) \otimes CH^{r-1}(X_2, 1) \to CH^r(X, 1) \to CH^r_{ind}(X, 1)$$

is uncountable.

Consider a sufficiently general product of three elliptic curves $X = E_1 \times E_2 \times E_3$. Let $X_1 = E_1$ and $X_2 = E_2 \times E_3$ in the setting of the theorem above. The Picard group is non trivial for an elliptic curve; $H^{1,0}(E_1) \neq 0$ and our result implies that the reduced regulator is nontrivial for a sufficiently general product of two elliptic curves. Hence we get

Corollary 4.3.2. *Let X be a sufficiently general product of three elliptic curves, then $CH^3_{ind}(X, 1)$ is uncountable.*

The same method to construct an indecomposable higher Chow cycle on a sufficiently general product of two elliptic curves is adopted to construct an indecomposable cycle for a product of four elliptic curves in [17]. However we have proved that such a cycle constructed using torsion points is decomposable. Torsion points are instrumental in constructing indecomposable cycles on elliptic curves. The analogy between rational curves on surfaces of the form $E_1 \times E_2$ and $K3$ surfaces and torsion points on elliptic curves, motivates us that indecomposable cycles can be constructed on the fourfold case, applying the philosophy of our construction. A higher Chow cycle can be constructed on the central fiber of the base space, but when we deform it to generic fiber it is no more a cycle. This approach is promising and work in progress.

Lastly we would like to present a result which follows from our main result and a corollary proved in a very recent paper. In [7], following corollary is proved;

Corollary 4.3.3 ([7][Corollary 1.1)**.** *] Let $X/\mathbb{C}$ be a very general member of a family of surfaces for which $H^1_{alg}(X, \Theta_X) \otimes H^{2,0}_v(X) \to H^{1,1}_v(X)$ is surjective. If the real regulator $r_{2,1} : CH^2(X, 1) \to H^{1,1}_v(X, \mathbb{R}(1))$ is nontrivial, then so is the transcendental regulator $\phi_{2,1}$.*

Let us describe the setting of this result briefly. It is assumed that X is a very general member of a family $\lambda : \chi \to S$ where χ and S are smooth quasiprojective varieties, λ is a smooth proper morphism and X corresponds to the central fiber. Then $H^1_{alg}(X, \Theta_X)$ is the image of the Kodaira-Spencer map $\kappa : T_0(S) \to H^1(X, \Theta_X)$, where Θ is the sheaf of holomorphic vector fields on X and $H^n_v(X)$ is the orthogonal complement of the fixed part of the monodromy group action of $\pi_1(S)$ on $H^n(X)$. This result is a corollary of [7][Theorem 1.1] which states that for a very general algebraic complex K3 surface, the transcendental regulator $\phi_{2,1}$ is nontrivial.

By our main result the real regulator $r_{2,1} : CH^2(X, 1) \to H^{1,1}_v(X, \mathbb{R}(1))$ is nontrivial for X a sufficiently general product of two elliptic curves. Also the surjectivity of the map $H^1_{alg}(X, \Theta_X) \otimes H^{2,0}_v(X) \to H^{1,1}_v(X)$ can be proved in a similar way to $K3$ surfaces given in [7].

So we get the following corollary;

Corollary 4.3.4. *Let X be a sufficiently general product of two elliptic curves, then the transcendental regulator $\phi_{2,1}$ is nontrivial.*

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